Double Default Correlation

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Abstract

Copula functions have become standard practice for pricing multi-name credit derivatives. Marginal default distributions are often chosen by using a simple deterministic intensity function. It is well-known that this approach only generates default time correlation and, apart from jumps due to default events, does not generate correlation between the conditional default intensities, or the conditional spreads. In this paper we consider pricing multi-name credit derivatives taking both default time correlation as well as default intensity correlation into account. This is achieved by defining two common factors, one for each type of correlation.

Further, we derive a fast way to price conditional on default events or survival for the factor model. Default and survival information is translated to information on the common factor. This approach allows us to graph conditional default intensities, or conditional CDS spreads, for simulated scenarios. These simulations show that our model results in a more realistic behavior of the conditional CDS spreads as one can distinguish both credit spread correlation as well as jumps in case of correlated default events.

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1 Introduction

Over the last decades the credit derivatives market has shown tremendous growth. The latest developments in this market can often be found in products whose payout depends on the default behavior

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of a portfolio of bonds or loans. Some examples are structured products such as CDOs, CLOs and CBOs, but also *n-th* to default swaps have gained popularity. The latest developments are the single tranche CDOs, or STCDOs, which involve the sale of a single CDO tranche to an investor. Throughout this paper we will refer to these products as multi-name credit derivatives. These developments have increased the need for pricing models which take default dependencies into account. In order to model default dependencies, a good starting point is the reduced form model, as this has become market practice for modelling single name credit derivatives such as credit default swaps, CDS. One can distinguish two methods to expand these reduced from models to allow for default correlation. The first and most obvious way is to model correlation between the dynamics of the default intensities of obligors directly. However, correlated Brownian motions do not seem to generate much default correlation and thus approaches focussing on jumps in intensities have become popular. Models uing this latter approach are often referred to as infectious default models and examples can be found in Davis and Lo (1999) and Jarrow and Yu (2001).

An alternative way to model default correlation is to impose a correlation structure through the use of copula functions. The advantage of this method is that one can first determine the marginal default distribution of every single obligor, after which the appropriate copula function can be chosen. The most widely used copula in practice is the Gaussian copula function, see for example Li (2000). Other copulas such as the t-copula have been used in Frey and McNeil (2001) and Mashal and Naldi (2002).

A disadvantage of the copula approach is that often Monte Carlo simulations are required in order to obtain prices for multi-name credit derivatives. Although this is not troublesome for determining a price, it will take a considerable amount of time to obtain accurate results when one wants to determine all spread sensitivities. Recent literature such as Andersen, Sidenius, and Basu (2003), Laurent and Gregory (2003) and Hull and White (2004), have focussed on the use of factor copulas, which are special cases of the general copula method. Using a one factor copula, the pricing of multi-name credit derivatives reduces to solving an integral numerically. The general idea is to condition on the realization of the common factor(s), after which all remaining sources of risk are independent. Thus after conditioning, one can obtain analytical expressions for the distribution of the number of defaults or the distribution of the loss of the entire portfolio. Integrating out the common factors is then the only remaining task.

A bridge between these two approaches for modelling default dependencies has been provided in the paper of Schönbucher and Schubert (2001). They show how to determine the dynamics of the default intensities implied by the chosen copula function, by conditioning on default and/or survival events. Among their findings, they show that the conditional default intensities jump at the moment of default of a correlated name. Apart from this jump effect, the copula model does not generate correlation between the processes for the conditional intensities. In practice, however, correlation between CDS spreads, and thus default intensities, is clearly present. Market-wide tightening or widenings of CDS spreads occur regularly. Furthermore, Duffie (2004) argues that incorporating both these types of correlation into one model is crucial for risk management of portfolios of corporate bonds as well as for pricing multi-name credit derivatives.

In this paper we show how we can extend the factor copula model to allow for credit spread correlation. We model default intensities as random processes, depending on a common factor and an idiosyncratic term. After conditioning on this common factor, the factor copula approach can be applied. This means that our model makes use of at least two common factors, one for modelling default event correlation and one for intensity correlation, or equivalently spread correlation. We will refer to the former common factor as the common copula factor, while the latter is referred to as the common intensity factor. In A recent paper, Yu (2004), both kinds of correlation are modelled, by directly modelling default intensities with correlated diffusions as well as jumps in case of defaults. His approach requires simulation techniques, simulating default events recursively. Another drawback of this approach is that it is hard to calibrate the marginal distributions such that the model is consistent with observed quotes in the CDS market. In the model described in this paper, calibration can be easily achieved using well-known techniques from short rate modelling, such as the extended Vasicek model described in Hull and White (1990).

Apart from providing pricing results for our model, we will show how spread dynamics can be determined conditional on default or survival information. Using Bayes rule, it is shown how conditional results can be obtained as fast as pricing under the proposed model. This fast method allows us to investigate the implications of our model on the behavior of conditional default intensities, or conditional CDS spreads. We plot the conditional CDS spread dynamics under different model assumptions, to investigate the effects of both intensity correlation, copula correlation or both.

The remainder of this paper is organized as follows: In the next section we will discuss the model. First the modelling of the intensity processes is discussed, which incorporates the intensity, or spread correlation by using a common factor. Next the section discusses copula functions. First the general case is considered, followed by a discussion on factor copulas. Finally the section will show how we can link both aspects into one model by first conditioning on the common intensity factor after which one can use the standard techniques used in factor copula models. In section 4 we will show how one can price products conditional on default and survival events, thus allowing us to investigate the dynamics of prices of multi-name credit derivatives, as well as the dynamics of the conditional default intensities. In section 5 some examples are shown, focussing on both the static implications as well as the dynamic implications of our double default correlation model. Finally, section 6 concludes.

2 The Model

The model will incorporate two kinds of correlation, default event correlation and default intensity correlation. The latter also implies CDS spread correlation, which can be observed in the market. The model is build upon two well-known ways to model default correlation. The first is to directly model correlation in the default intensities of the underlying obligors. The second method to create default correlation is by means of copula functions. Here we combine both these methods into one model. We will first discuss both methods separately, after which we will show how they can be combined.

2.1 Intensity model

Single name default modelling can be divided in two approaches. The first method to model credit risk is known as the structural approach. In this approach, due to Merton (1974), the asset and liability structure of an obligor is explicitly modelled and default is triggered in case the obligor can not repay its outstanding debt at maturity. Other authors have extended this approach in a number of ways, but models became complicated as the required input can be extremely hard to obtain. Furthermore these model can be hard to calibrate to observed prices of defaultable bonds, or CDSs.

Later papers, for instance Jarrow and Turnbull (1995), Lando (1998) and Duffie and Singleton (1999) consider a different approach known as the reduced form method, or intensity based method. As the name suggests, a default intensity is modelled, which determines the default probability over a small period of time. These models can easily be calibrated to observed credit spread term structures, which among with their ease of use and their close relationship to interest rate models, have caused them to become market practice for the pricing of single name credit derivatives.

In this paper we build up default intensities from different independent processes. To illustrate the model we assume that the processes are mean reverting of the Vasicek (1977) type. By means of a common factor, the default intensity processes of the different names are correlated. The process for the default intensity of name i, λ_i, is given by:

$$\begin{aligned} d\lambda_i &= \tilde{\rho}_i dh_0 + dh_i \\ dh_j &= \kappa_j \left(\theta_j - h_j\right) dt + \sigma_j dW_j \qquad j = 0, i \end{aligned} \tag{1}$$

Here, all Brownian motions, dW_j are independent. Thus every intensity process is build from two independent Vasicek type processes, h_0 and h_i. We will refer to h_0 as the common intensity factor and this should not be confused with the common copula factor, introduced later in the paper. The parameter $\tilde{\rho}_i$ gives the weight of the common intensity factor for name i and thus determines the correlation between the intensities of the different names.[1] The tilde in $\tilde{\rho}_i$ is introduced to distinguish

[1] *The parameter $\tilde{\rho}_i$ determines the amount of correlation between intensities, but it should not be seen as a correlation in the sense that it falls in the interval $[-1, 1]$. One can determine the actual correlation between intensities:* $\frac{\sigma_0^2 \rho_i \rho_j}{\sqrt{\rho_i^2 \sigma_0^2 + \sigma_i^2}\sqrt{\rho_j^2 \sigma_0^2 + \sigma_j^2}}$.

it from the copula correlation parameter ρ_i, which will be defined later. The terms h_i can be regarded as idiosyncratic risk sources as these only affect the credit-worthiness of a single obligor. The choice of Vasicek type processes is for illustrative purposes and one is not restricted to this choice; alternative processes for h_i can be used as well. For instance, one can use square root processes for the h_i, as in Cox, Ingersoll, and Ross (1985) in order to prevent the possibility of negative default intensities. Alternatively one can use an extended Vasicek process for the h_i, $i = 1, \ldots, N$ as described in Hull and White (1990), in case one wants to calibrate to a term structure of observed CDS quotes for the individual names. Calibration is also possible by means of deterministic functions for the idiosyncratic terms h_i, such as a piecewise constant function. In case one specifies h_0 as a jump diffusion, the default intensities can display sudden market-wide increases or decreases in intensities, or CDS spreads, not caused by default events of names in the basket. Idiosyncratic jumps in default intensities can easily be modelled by allowing for jumps in the processes h_i, $i = 1, \ldots, N$.

We denote the default probability for name i up to time T by $p_i(T)$. For any choice of processes h_i we can write the default probability for name i as follows:

$$\begin{aligned} p_i(T) &= 1 - \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^T \lambda_i(s)\, ds\right)\right] \\ &= 1 - \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\tilde{\rho}_i \int_0^T h_0(s)\, ds\right)\right] \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^T h_i(s)\, ds\right)\right] \end{aligned} \tag{2}$$

Here, $\mathbb{E}^{\mathbb{Q}}$ denotes the expectation operator under the risk-neutral measure $\mathbb{Q}$. The second line of (2) is obtained by using the independence of the processes h_i. The final expectation can usually be obtained directly from results of short rate modelling literature, due to the close relationship between survival probabilities and zero bond prices. For affine jump diffusions, one can use the techniques described in Duffie, Pan, and Singleton (2000).

This setup allows us to determine multivariate default probabilities. Therefore we first need to condition on the integral over the common factor, $Z(T) \equiv \int_0^T h_0(s)\, ds$. In case this is done, the conditional default probabilities are given by:

$$\begin{aligned} p_i(T|\, Z(T) = z) &= 1 - \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^T \lambda_i(s)\, ds\right)\middle|\, Z(T) = z\right] \\ &= 1 - \exp(-\rho_i z)\, \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^T h_i(s)\, ds\right)\right] \end{aligned} \tag{3}$$

After conditioning, all default probabilities are independent. This allows for easy determination of the multivariate default probabilities. Conditional on the common intensity factor we can thus determine the probability of exactly n defaults, for $n = 0, \ldots, N$. The methods that can be used to achieve this are the same as those that can be used for factor copula models and will be discussed later in the paper. One can then integrate z in (3) out, using its distribution to obtain unconditional default probabilities. In our example every process h_i is of the Vasicek type, thus the term $Z(T)$ has a normal distribution. In Appendix A it is shown how one can derive its mean and variance.

In the multivariate default modelling approach discussed in the next subsection, we will ignore the marginal default distribution derived in here. We consider the marginal default probabilities as input and assume, for the moment, that these are independent.

2.2 Copula functions

Correlating default intensities will often not give the desired level of correlation between the default events, see for example Schönbucher (2003). By using copula functions this problem can be overcome as these directly model the correlation between the default times, rather than the correlation between the intensities. The main advantage of the copula approach is that one can easily impose correlation structures, without altering the marginal distributions of the underlying default times. This is one of the main reasons for its popularity in default correlation modelling, as practitioners want to use the marginal distributions for the underlying names which have been calibrated to observed market quotes for credit default swaps. Thus one can first calibrate to the CDS market after which the copula framework can be used to impose default time correlation. The resulting model is then, by construction, consistent with the observed quotes from the CDS market.

An N-dimensional copula function specifies the joint distribution of N variables, each with a standard uniform distribution. Thus in case we have N uniformly distributed variables U_i, a copula function can be seen as the following joint distribution:

$$C(u_1, u_2, \ldots, u_N) = \Pr(U_1 \leq u_1, U_2 \leq u_2, \ldots, U_N \leq u_N) \tag{4}$$

For any random variable X_i with distribution function F_i we know that $F_i(X_i)$ has a uniform distribution. This relation allows us to transform any random variable to a uniform variable. In case we have a joint distribution function $F(x_1, x_2, \ldots, x_N)$ we can write this using a copula function:

$$C(F_1(x_1), F_2(x_2), \ldots, F_N(x_N)) = F(x_1, x_2, \ldots, x_N) \tag{5}$$

A good reference, discussing copula functions in more detail is Nelsen (1999). In practice the Gaussian copula is the most widely used copula, see for instance Li (2000). The Gaussian copula is defined as follows:

$$C(\mathbf{u}) = \Phi_N\left(\Phi^{-1}(u_1), \Phi^{-1}(u_2), \ldots, \Phi^{-1}(u_N), \Sigma\right), \tag{6}$$

where Σ denotes the N x N correlation matrix and Φ the cumulative normal distribution function. $\Phi_N(\ldots, \Sigma)$ denotes the N dimensional normal distribution function with correlation matrix Σ. When modelling default events, it makes sense to regard the values U_i as being trigger levels. At time T obligor i will have defaulted in case $p_i(T) > U_i$, where $p_i(T)$ denotes the default probability of name i up to time T. The copula function then defines the joint distribution of the trigger levels U_i.

In case a general copula function is used, one is often forced to obtain prices and hedge parameters by means of Monte Carlo simulation. Although this is often easy to implement, the main disadvantage of this approach is that it is computationally intensive, especially when one wants to determine the hedge parameters. Much faster computation can be achieved in case we restrict ourselves to factor copulas, which will be discussed next.

2.2.1 One-Factor Copula

The idea behind the one factor copula is similar to that behind the general copula, as defaults are triggered by correlated random variables. Correlation between these random variables is now modelled by using a one factor model. Here we shall give a brief discussion on factor copulas for pricing multi-name credit derivatives, based on Hull and White (2004).

We have $N+1$ independent random variables Y_i, $i = 0, \ldots N$, each having zero mean and unit variance. We assume that Y_0 denotes the common factor and Y_i the idiosyncratic factor for name i. Instead of the uniform trigger levels used before, we now define our trigger levels X_i as follows:

$$X_i = \rho_i Y_0 + \sqrt{1-\rho_i^2} Y_i, \tag{7}$$

In this specification, ρ_i denotes the correlation parameter for name i. Note that the correlation is with respect to the common factor and thus the correlation between two trigger levels for names i and j is given by $\rho_{i,j} = \rho_i \rho_j$. The distribution function of X_i is denoted by F_i and that of the variables Y_i by H_i. Thus F_i depends on the choice of the distributions for both Y_0 and Y_i, as well as the correlation parameter ρ_i. Obviously, the factor model is more restrictive than the general copula function and one might want to choose the correlation parameters, ρ_i such that the model correlation matches the desired input correlation as close as possible. One can perform an optimization routine, such as in Andersen, Sidenius, and Basu (2003), to obtain the correlation parameters.

We say that a default has occurred by time t, in case a certain variable, $\chi_i(t)$, has crossed the trigger level X_i In order to match the marginal default probabilities exactly these variables need to be chosen appropriately:

$$\begin{aligned} p_i(T) &\equiv \Pr(\tau_i \le T) = \Pr(X_i \le \chi_i(T)) \Longleftrightarrow \\ \chi_i(T) &= F_i^{-1}(p_i(T)). \end{aligned} \tag{8}$$

One can also view $\chi_i(t)$ as a process and default occurs the moment this process crosses the trigger level X_i. The advantage of using the one-factor copula becomes clear when one conditions on the common factor. Once this value is known, all remaining sources of risk are independent, thus allowing for easy calculations. The default probabilities conditional on the realization of the common factor are given by:

$$p_i(T|Y_0) \equiv \Pr(\tau_i \le T|Y_0) = H_i\left(\frac{\chi_i(T) - \rho_i Y_0}{\sqrt{1-\rho_i^2}}\right), \tag{9}$$

Thus when performing calculations, one can first condition on the common factor and then integrate it out, using the distribution of the common factor.

2.3 Intensity and Copula correlation

Although one can obtain a large amount of default correlation by means of copula functions, one also might want to allow for correlated default intensities. In practice one can regularly observe market-wide spread increases or decreases, which can not be explained by the copula framework applied to uncorrelated default intensities, unless these moves are caused by default events. By including intensity correlation in our copula framework we can have closely correlated default intensities as well as large default correlation caused by the copula structure.

In order to combine the intensity correlation and the one-factor copula correlation in the model, we note that the common factor for the intensities and the one for the copula structure are independent. The idea is to first condition on the common factor for the default intensities, Z. After that the same approach as described before can be used.

$$\begin{aligned} \chi_i(T|Z) &= F_i^{-1}(p_i(T|Z)). \\ p_i(T|Z, Y_0) &\equiv \Pr(\tau_i \le T|Z, Y_0) = H_i\left(\frac{\chi_i(T|Z) - \rho_i Y_0}{\sqrt{1-\rho_i^2}}\right) \end{aligned} \tag{10}$$

After knowing the conditional default probabilities, $p_i(T|Z, Y_0)$ for all names, all remaining sources of risk are independent. Using one of the methods described later in this paper, one can determine the conditional value of a credit derivative: $V(Z, Y_0)$, from the conditional default probabilities. Finally both common factors need to be integrated out, first over the distribution of the common copula factor, Y_0, followed by integration over the distribution of the common intensity factor, Z:

$$V = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} V(z, y)\, dH_0(y)\, dG(z) \tag{11}$$

where G denotes the distribution of Z. In a similar manner, one can determine the hedge parameters of the credit derivative with respect to marginal probabilities $p_i(T)$, or with respect to CDS spreads. Again one can use numerical techniques such as Gaussian quadratures to solve the integrals numerically.

Although we have described our model in its simplest form, one can easily extend it to allow for additional features. In a recent paper by Andersen and Sidenius (2004) extensions to the standard case are derived by allowing for factor dependent random recovery rates as well as factor dependent random factor loadings. This can easily be incorporated in the model presented here. Moreover, one can allow for a recovery rate depending on either the common copula factor, the common intensity factor or both. The same can be said for the factor loadings.

One can also allow for correlation between the two common factors. Negative correlation might be desirable. For instance, if the economy is in a bad state with a lot of defaults occurring, then the realization of the common copula factor is likely to be very low. With negative correlation this means that the common factor for the intensities will likely be high, thus meaning that intensities which have a large weight for the common intensity factor, are also likely to be high. This means that we can model correlation such that a scenario with a lot of defaults, is accompanied by market wide increases of the intensity. Thus we can have companies with high intensities in a bad state of the economy, but without the jumps in case of defaults.

3 Pricing Multi-name Credit Derivatives

When pricing multi-name credit derivatives such as CDOs, one needs to determine the distribution of the cumulative loss on the portfolio. In case of a homogenous portfolio it suffices to derive the distribution of the number of defauls. After conditioning on the realization of both common factors, the remaining sources of risk are independent. In order to determine the loss distribution conditional on the realization of the common factors efficiently, some algorithms have been proposed. Laurent and Gregory (2003) propose to use Fourier transforms, while Andersen, Sidenius, and Basu (2003) and Hull and White (2004) propose to use recursive algorithms, for either the loss distribution, or the distribution of the number of defaults. The grid based method of Andersen, Sidenius, and Basu (2003) is intuitively appealing and allows for straightforward determination of the hedge parameters.

3.1 Basket Default Swaps

After applying one of the methods discussed above, one can determine the conditional distribution of the numer of defaults. One can use a simplified version of the method of Andersen, Sidenius, and Basu (2003) which can be seen as a recombining binomial tree. This tree has N steps, each step corresponding to one name in the basket. At time step i the probability of an up-move will be $p_i\left(T|Z,Y_0\right)$ and thus the probability of a down move will be $1-p_i\left(T|Z,Y_0\right)$. Every up-move corresponds to a default event, while a down-move corresponds to survival. Working from the root of the tree back to the end values, one obtains the probability of exactly n defaults. To illustrate this, let $\delta_{i,j}$ denote the path probability of step i and height j. We assume that i runs from $0,\ldots,N$, and $j=0,\ldots,i$. Thus we start the algorithm by setting $\delta_{0,0}=1$, after which the algorithm continues as follows:

$$\delta_{i,j}=\begin{cases}(1-p_i)\cdot\delta_{i-1,j} & if \quad j=0\\ p_i\cdot\delta_{i-1,j-1}+(1-p_i)\cdot\delta_{i-1,j} & if \quad 0<j<i\\ p_i\cdot\delta_{i-1,j-1} & if \quad j=i\end{cases}. \tag{12}$$

For brevity we have used $p_i=p_i\left(T|Z,Y_0\right)$. If one is done working through the tree, the values $\delta_{N,j}=\delta_{N,j}\left(T|Z,Y_0\right)$ give the conditional probability of exactly j defaults in the basket of names. Integrating these terms out over the distributions of both common factors will result in the probability of exactly j defaults, for $j=0,\ldots,N$. Now let:

$$\begin{aligned}\pi\left(j,T\right) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta_{N,j}\left(T|Z,Y_0\right)dH_0\left(y\right)dG\left(z\right) \qquad (13)\\ P\left(n,T\right) &= \sum_{j=n}^{N}\pi\left(j,T\right)\end{aligned}$$

Thus $\pi\left(j,T\right)$ denotes the unconditional probability of exactly j default events by time T, while $P\left(n,T\right)$ denotes the probability of at least n defaults by time T. As an example we give the pricing formula for a first to default swap with premium payments c, assuming all names have equal recovery rate, R:

$$\begin{aligned}V_{1st} &= (1-R)\sum_{j=1}^{M}\left(P\left(1,T_j\right)-P\left(1,T_{j-1}\right)\right)e^{-r\frac{1}{2}\left(T_j+T_{j-1}\right)} \qquad (14)\\ &\quad -c\sum_{j=1}^{M}\delta\left(T_{j-1},T_j\right)\left(\left(1-P\left(1,T_j\right)\right)e^{-rT_j}+\frac{1}{2}\left(P\left(1,T_j\right)-P\left(1,T_{j-1}\right)\right)e^{-r\frac{1}{2}\left(T_j+T_{j-1}\right)}\right)\end{aligned}$$

Here T_i denote the timings of the premium payments and δ denotes the appropriate daycount fraction. We have assumed a fixed interest rate r, but this can easily be replaced by more general discounting functions. The first summation in (14) determines the payoff leg in case of default, while the second summation gives the expected net present value of the premium payments. It is assumed that defaults occur halfway a period and in case of default the protection buyer has to pay accrued premium, while receiving the loss given default, $(1-R)$.

3.2 CDOs

For valuing CDO tranches a different approach should be taken, based on the expected loss on the tranche. By taking the threshold levels, L^- and L^+, into account one can easily determine the conditional expected loss on the tranche, using one of the methods discussed above. Integrating out over both common factors will yield the unconditional expected loss on the tranche. This can be repeated for a number of different points in time. Let these values be denotes by $Eloss\,(T)$. The value of a CDO tranche paying coupon of c will then be:

$$\begin{aligned} V_{CDO} &= (1-Eloss\,(T))\,e^{-rT} \\ &\quad +c\sum_{j=1}^{M}\delta\,(T_{j-1},T_j)\left(1-\frac{1}{2}\frac{Eloss\,(T_{j-1})+Eloss\,(T_j)}{L^+-L^-}\right)e^{-rT_j} \end{aligned} \tag{15}$$

Here we have assumed that coupon payments are made over the remaining size of the tranche.

The model also allows for recovery rates being dependent on one or both of the common factors. One can take this into account when determining the conditional expected loss. After integrating out over both common factors, one obtains the unconditional expected loss and formula (15) can be applied to obtain the price of the CDO tranche.

4 Conditional Default Probabilities

The proposed model can be used for pricing a certain multi-name credit derivative at initialization. However, in case one wants to see the behavior of the product price over time, one needs to condition on the available information. This information can consist of default events or survival, as well as information on the default intensities. The paper by Schönbucher and Schubert (2001) shows how one can take available information into account when modelling the dependency structure using a copula function, by deriving the process for the conditional default intensities. These conditional default intensities should not be confused with the unconditional intensities, λ_i. The latter determine the marginal default distributions, while the conditional intensities are the actual default intensities, implied by the copula function, taking all available information into account. For general copula functions, the formulas derived in Schönbucher and Schubert (2001) are hard to implement and will take up a large amount of computational time. When using the factor copula, one can translate default and survival information directly to information on the common factor. As more information is revealed, more is known about the value of the common factor, thus changing its distribution.

Throughout this section we ignore intensity correlation, for simplicity. It is straightforward to use the methods described here in case of both intensity and copula correlation as one can first condition on the common factor for the intensity correlation, after which the formulas discussed in this subsection can be applied.

4.1 Information translated to the copula variables

As was shown in formula (7) of the previous section, every name has a copula variable, X_i, constructed from a common factor and an idiosyncratic term. A default event has occurred by time T in case this variable was smaller than the barrier $\chi_i(T) = F_i^{-1}(p_i(T))$. One can thus determine the exact default time τ_i by solving $\chi_i(\tau_i) = X_i$. Alternatively, one can specify $u_i = F_i(X_i)$, which has a uniform distribution and a default event is triggered as soon as $1 - \exp\left(-\int_0^t \lambda_i(s)\,ds\right) > u_i$.[2]

Due to this structure of the model, one can easily translate survival information to information on the copula variable underlying that name. Suppose current time is t and we know that name i has survived up to time t. Similar as in Rogge and Schönbucher (2003), we can translate this to the copula variable X_i as follows:

$$\begin{aligned} \tau_i &> t \iff \\ u_i &> 1 - \exp\left(-\int_0^t \lambda_i(s)\,ds\right) \iff \\ X_i &> F_i^{-1}\left(1 - \exp\left(-\int_0^t \lambda_i(s)\,ds\right)\right). \end{aligned} \tag{16}$$

At time t, the integrals $\int_0^t \lambda_i(s)\,ds$ are known, so here is no randomness in (16). Similarly for default at time t we get:

$$\begin{aligned} \tau_i &= t \iff \\ u_i &= 1 - \exp\left(-\int_0^t \lambda_i(s)\,ds\right) \iff \\ X_i &= F_i^{-1}\left(1 - \exp\left(-\int_0^t \lambda_i(s)\,ds\right)\right). \end{aligned} \tag{17}$$

[2] *Equivalently, one can also define a default event in case* $\exp\left(-\int_0^t \lambda_i(s)\,ds\right) < 1 - u_i$. *Due to the uniform distribution one can replace* $1 - u_i$ *by* $\tilde{u}_i$ *another uniform distribution.*

So we see that survival information results in knowing that the copula variable of the name under consideration is larger than a certain value. Knowing the default time of a certain name, results in knowledge of the exact value of the latent copula variable X_i, of the corresponding name.

4.2 Density of the common factor

The next step is to use the available default and survival information to derive the conditional density of the common factor. First we need to introduce some filtrations:

Definition 1 *:*

1. *The background filtration $\mathcal{G}_t$ denotes the information generated by all $h_i(t)$*
2. *Filtration $\mathcal{F}_t^i$ is the augmented filtration that is generated by $N_i(t)$.*
3. *Filtration $\mathcal{H}_t^i$ reflects information on default of name i and information on the default intensities up to time t. Thus $\mathcal{H}_t^i = \sigma\left(\mathcal{F}_t^i \cup \mathcal{G}_t\right)$*
4. *Filtration $\mathcal{H}_t$ contains default information about all names up to time t, in combination with information on all default intensities up to time t. Thus $\mathcal{H}_t = \sigma\left(\bigcup_i \mathcal{H}_t\right)$.*

We are interested in calculating the default probabilities conditional on available information. First we start by considering the risk neutral probability of default conditional on $\mathcal{G}_t$, for a certain name i. Thus the intensity processes for all names are known up to time t. We get:

$$\begin{aligned} \Pr\left(\tau_i \leq T|\mathcal{G}_t\right) &= 1 - \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^T \lambda_i(s)\,ds\right)|\mathcal{G}_t\right] \\ &= 1 - \exp\left(-\int_0^t \lambda_i(s)\,ds\right)\mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T \lambda_i(s)\,ds\right)|\mathcal{G}_t\right]. \end{aligned} \tag{18}$$

The expectation in (18) can easily be determined since λ_i is driven by two Vasicek processes and $h_i(t)$ is known under $\mathcal{G}_t$, as is $h_0(t)$.

We next determine the process which triggers defaults, which is useful for the rest of our analysis. Again this is chosen such that the default probabilities under the copula model will match those resulting from the marginal distributions.

$$\chi_i\left(T|\mathcal{G}_t\right) = F_i^{-1}\left(\Pr\left(\tau_i \leq T|\mathcal{G}_t\right)\right). \tag{19}$$

We need to evaluate this function at both t and T, so for brevity we shall denote these by:

$$\begin{aligned} \chi_i^t &= \chi_i \left(t | \mathcal{G}_t \right) \\ \chi_i^T &= \chi_i \left(T | \mathcal{G}_t \right). \end{aligned} \tag{20}$$

From (18) one can see that χ_i^t involves no expectation. Furthermore it is chosen such that $\tau_i > t \Longleftrightarrow X_i > \chi_i^t$. The second term in (20) can be seen as the barrier for default before time T, taking intensity information up to time t into account. Thus when intensities up to time t are known, default for name i occurs before time T when $X_i \leq \chi_i^T$. Obviously, at time t we would have noticed a default event and thus we should condition on $\mathcal{H}_t^i$ as well. In case the name under consideration has defaulted by time t default probability calculations for that name will be trivial and thus we only consider the case of survival. Thus the next step is to condition on the fact that the name has survived up to time t and translate this to the latent copula variables:

$$\begin{aligned} & \Pr\left(\tau_i \leq T | \mathcal{H}_t^i \right) \\ = & \Pr\left(X_i \leq \chi_i^T \middle| X_i > \chi_i^t \right) \\ = & \frac{\Pr\left(\chi_i^t < X_i \leq \chi_i^T \right)}{\Pr\left(X_i > \chi_i^t \right)} = \frac{F_i\left(\chi_i^T \right) - F_i\left(\chi_i^t \right)}{1 - F_i\left(\chi_i^t \right)} \end{aligned} \tag{21}$$

Finally we can condition on the realization of the common factor: $Y_0 = y$:

$$\begin{aligned} & \Pr\left(\tau_i \leq T | \mathcal{H}_t^i \wedge \{Y_0 = y\} \right) \\ = & \frac{H_i\left(\frac{\chi_i^T - \rho_i y}{\sqrt{1-\rho_i^2}} \right) - H_i\left(\frac{\chi_i^t - \rho_i y}{\sqrt{1-\rho_i^2}} \right)}{1 - H_i\left(\frac{\chi_i^t - \rho_i y}{\sqrt{1-\rho_i^2}} \right)}, \end{aligned} \tag{22}$$

which easily follows from the distribution of the idiosyncratic part of the copula variable X_i, denoted by H_i. Further note that the same results are obtained when conditioning on the more general filtration $\mathcal{H}_t$ instead of $\mathcal{H}_t^i$ and $\mathcal{G}_t$, because the additional information from the other names relevant for name i is already incorporated in the realization of the common copula factor: $Y = y$.

To determine the probability unconditional on the realization of the common factor, we need to integrate the common factor out, using its density. Doing this, we need to keep track of available information in $\mathcal{H}_t$, since one can not use the initial distribution of the common factor anymore. To determine the default probability of name i given all available information at time t we get:

$$\Pr\left(\tau_i \le T|\mathcal{H}_t\right) \quad (23)$$
$$= \int_{-\infty}^{\infty} \Pr\left(\tau_i \le T|\mathcal{H}_t \wedge \{Y_0 = y\}\right) \; h_0\left(y|\mathcal{H}_t\right) dy.$$

The function $h_0\left(y \mid \mathcal{H}_t\right)$ denotes the conditional density function of the common factor. Note that the probability term in the integration is independent for all names, since we have conditioned on the realization of the common factor. To determine the unconditional distribution of the number of defaults from here on, one can apply the exact same approach as usual for factor copula models. There are two differences, however: first the conditional probabilities are now conditional on both the common factor as well as default and survival information, i.e. we use $\Pr\left(\tau_i \le T|\mathcal{H}_t \wedge \{Y_0 = y\}\right)$. Second, we do not have the original distribution for the common factor, as we have conditioned on default and survival information up to time t through $\mathcal{H}_t$, as well as intensity information through $\mathcal{G}_t$, which is included in $\mathcal{H}_t$.

So the problem has been reduced to determining the density of the common factor, conditional on default and survival information as well as information on the default intensities. To determine this density we make a small adjustment of threshold value defined in (19), as we now also conditional on $\mathcal{F}_t^i$, which captures information on the default or survival of the name.

$$\tilde{\chi}_i^t \equiv \tilde{\chi}_i\left(t|\mathcal{F}_t^i \cup \mathcal{G}_t\right) \equiv F_i^{-1}\left(1 - \exp\left(-\int_0^{\min(t,\tau_i)} \lambda\left(s\right) ds\right)\right). \quad (24)$$

Note that we now have $\min\left(t, \tau_i\right)$ as the upperbound for integration, thus this definition can be used for both survivors as well as defaulted names. We only need to evaluate this for time t so there is no remaining uncertainty. Further we define $\mathcal{D}_t$ and $\mathcal{S}_t$ to be the set of defaulted names and survivors at time t, respectively. Now, to determine the conditional density of the common factor, we use Bayes theorem:[3]

[3] *We have used rather informal notation in an attempt to simplify the formula. The difficulty comes from the fact that both equality as well as inequality signs occur in the available information. One should read* $\Pr\left(X = x\right) = f\left(x\right) dx$*, where* f *denotes the density of* X*.*

$$
\begin{aligned}
&\Pr\left(Y=y|\mathcal{F}_t\cup\mathcal{G}_t\right) \qquad (25)\\
&\Pr\left(Y=y|\bigcap_{i\in\mathcal{D}_t}\{x_i=\tilde{\chi}_i^t\}\cap\bigcap_{i\in\mathcal{S}_t}\{x_i>\tilde{\chi}_i^t\}\right)\\
=\ &\frac{\Pr\left(\bigcap_{i\in\mathcal{D}_t}\{x_i=\tilde{\chi}_i^t\}\cap\bigcap_{i\in\mathcal{S}_t}\{x_i>\tilde{\chi}_i^t\}\mid Y=y\right)\cdot\Pr\left(Y=y\right)}{\Pr\left(\bigcap_{i\in\mathcal{D}_t}\{x_i=\tilde{\chi}_i^t\}\cap\bigcap_{i\in\mathcal{S}_t}\{x_i>\tilde{\chi}_i^t\}\right)}.
\end{aligned}
$$

The first term in the numerator can easily be determined using the independence property after conditioning. The other term in the numerator is just the (unconditional) density of the common copula factor. Finally the denominator can be obtained by integrating the term in the numerator over the density of the common factor. For implementation purposes, it is important to note that the numerical integration needed for determining the denominator needs to be performed only once for a given information set, since it does not depend on the value of the common factor.

Integrating probabilities out over the conditional density of the common factor can be done using standard numerical integration techniques, such as Gaussian quadratures.

Using this approach one can determine default probabilities and thus prices of multi-name credit derivatives conditional on default and survival information, as implied by the copula model.

5 Examples

The model derived in this paper allows one to incorporate both kinds of correlation when pricing multiname credit derivatives. In this section the effects of both correlation parameters on the prices of CDO tranches will be investigated. Furthermore we shall investigate the conditional behavior of CDS spreads using simulation. Simulation results show possible paths for the conditional 5 year CDS spreads.

5.1 Static implications of the double correlation model

In various papers it was already argued that intensity correlation does not generate enough default correlation. In the double correlation model this should be reflected through a small dependency on the intensity correlation parameter. Here we determine the fair coupon of 5 different tranches of a CDO on 100 equally weighted names, all with a recovery rate at 50%. The threshold levels for the tranches are set at 3%, 6%, 9%, 12% and 22%. Furthermore we rewrite the intensity processes in (1) to look like:[4]

[4] Note that this is still in the same form as (1) as one can adjust the process parameters for h_i.

$$\lambda_i = \tilde{\rho}_i h_0 + (1 - \tilde{\rho}_i)\, h_i \tag{26}$$

Where all h_i are of the Vasicek type with mean reversion speed 0.3, mean reversion level 150 basispoints, volatility parameter 50 basispoints and initial values of 150 basispoints. The parameters $\tilde{\rho}_i$ are all equal to the intensity correlation, ξ. For both processes we choose their respective volatility parameter such that the volatility of the intensities is equal to the 50 basispoints.

		$\xi = 0\%$	$\xi = 25\%$	$\xi = 50\%$	$\xi = 75\%$	$\xi = 100\%$
	T1	1921	1903	1886	1868	1851
	T2	530	541	552	562	572
$\rho^2 = 0\%$	T3	15	21	27	33	40
	T4	0	0	0	0	0
	T5	0	0	0	0	0
	T1	1624	1614	1605	1595	1585
	T2	603	603	604	604	605
$\rho^2 = 10\%$	T3	172	176	181	185	190
	T4	43	46	49	52	54
	T5	4	4	5	5	6
	T1	1324	1318	1312	1306	1300
	T2	575	574	573	572	571
$\rho^2 = 25\%$	T3	274	276	277	279	280
	T4	138	139	140	142	143
	T5	37	28	39	40	41
	T1	936	934	932	930	927
	T2	496	491	487	485	483
$\rho^2 = 50\%$	T3	302	306	309	311	312
	T4	203	205	206	207	208
	T5	106	107	107	107	108

Table 1: *Effect of both default time correlation (rows, ρ^2) as well as intensity, or spread correlation (columns, ξ) on the prices of 5 different tranches. Thresholds for the tranches are 3%, 6%, 9%, 12% and 22%.*

The table above shows the fair premiums for the 5 different CDO tranches for different levels of copula correlation and intensity correlation. Results are obtained using numerical integration according to the Gauss-Legendre scheme with 96 evaluation points, see Abramowitz and Stegun (1972) for more details. From the table one can see that the effect of intensity correlation is indeed small, but it can not be ignored. The effect of copula correlation is obviously much larger. The effect of both type of correlations is similar in that the fair coupon for the Equity tranche decreases as either default time, or default intensity correlation increases. For more senior tranches, increasing either type of correlation results in an increase in fair premium. Furthermore, there seem to be some cross effects. From the table one can clearly see that the effect of intensity correlation is larger in case copula correlation is small.

5.2 Dynamic implications of the double correlation model

As shown in Schönbucher and Schubert (2001), conditional default intensities will jump upwards in case a (positively) correlated name defaults, when using copula functions. In case no default occurs, the conditional default intensities are likely to decrease over time. Imposing intensity correlation will not result in jumps, but can generate correlation between the spread of different names. In order to investigate the dynamic behaviour more carefully, we perform some simulations for CDS quotes for different names with a five year tenor.[5] We generate scenarios over ten years taking daily steps. At every time step the fair spread of a five year CDS is determined for every name, while taking all available information at that time into account. This allows us to investigate model implied behaviour for five year CDS spreads. Moreover, we can take a closer look at the effects of copula correlation as well as intensity correlation in combination with survival and default events.[6] Using our proposed method, all required calculations can be performed fast.

Throughout this section we show the processes for the CDS quotes for three names out of a larger basket. Only scenarios are chosen where none of the three names default. Parameters are chosen such that the first two names have intensity processes of the Vasicek type, with mean reversion speed 0.3, mean reversion level of $150BP$ and volatility at $50BP$. The third name has a flat unconditional intensity at $50BP$ and mainly functions to investigate the effect of copula correlation.

We first consider a situation without copula correlation, thus a default event will not affect the other names. That is, the unconditional results are equal to the conditional results. In the two figures below, we show the simulated CDS spreads for the three names. In the figure on the left-hand-side, the correlation between the intensities for the first two obligors is equal to zero, while it is set at 90% for the figure on the right hand side.

[5] *Although this section only considers the dynamics of the fair spread of 5 year CDSs, one can alternatively derive the dynamics of the conditional default intensities. Due to the close relationship with CDS quotes, the results will be very similar.*

[6] *For simulation we have assumed that there is no risk premium. Thus we simulate under the pricing measure*

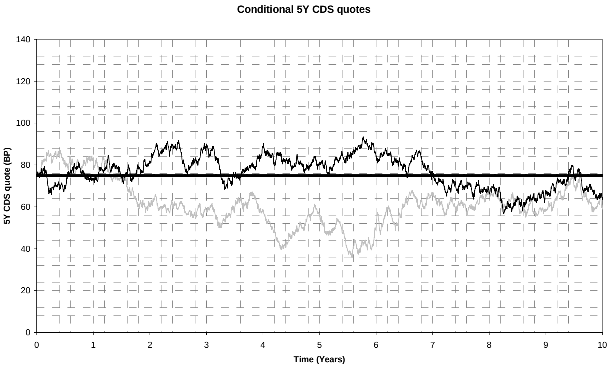

Figure 1: Simulated unconditional default intensities for 3 names. The intensities for the first two names follow a Vasicek type process, the third intensity is constant at 150BP.

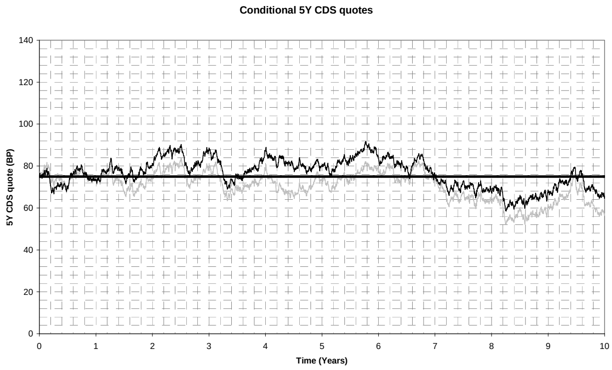

Figure 2: Resulting conditional 5Y CDS spreads.

In both figures one can clearly see mean-reverting behaviour of the CDS prices. This is a direct result of the mean reversion of the default intensities. When comparing figures 1 and 2 one can clearly see the effect of the intensity correlation on the CDS spreads. Due to the large intensity correlation of 90%, both spreads move closely together.

We next introduce copula correlation in the model and investigate the effects, both with and without intensity correlation. Due to the copula structure, default events as well as survival will affect the default probabilities of the surviving names and thus their CDS spreads. As discussed in section 4, a default event can change the conditional distribution of the common copula factor dramatically, thus resulting in a large jump effect on the CDSs on surviving names. Also survival will bring in additional information about the level of the common copula factor, however this will not lead to a large change in available information.

The figures below show the conditional dynamics of CDSs on the three different names in case copula correlation is introduced. Again two names have random unconditional intensity processes of the Vasicek type, while the default event for the third name is modelled by means of a constant unconditional intensity. The figure on the left-hand side does not incorporate intensity correlation between the first and second obligor, while the figure on the right hand side does. The same random numbers as in figures 1 and 2 were used. In both figures copula correlation is set at 20%.

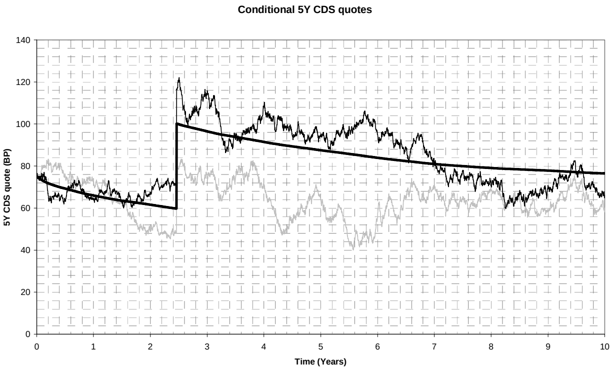

Figure 3: Conditional 5Y CDS spreads for a basket consisting of 10 names.

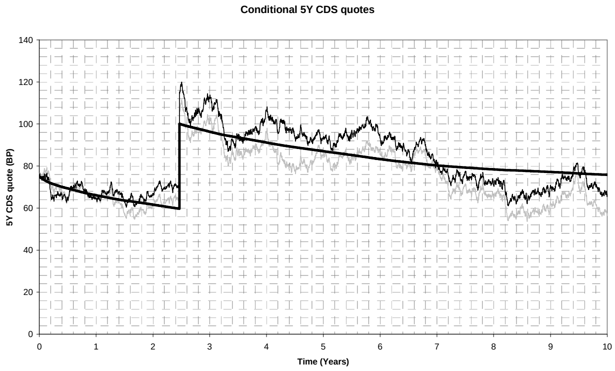

Figure 4: Conditional 5Y CDS spreads for a basket consisting of 10 names.

In both figures one can clearly see the effect of a default event. After about 2.5 years, one name in the basket defaults and thus there is a jump in the conditional intensities of the surviving names. This jump in conditional intensities leads directly to a jump in the conditional CDS spreads, which jump from a level of about 60 basispoints to about a 100 basispoints. The size of this jump is determined by both the copula correlation as well as the timing of the default event. In case the first default event would have occurred earlier, the jump in the intensities of the surviving names would have been even larger. Further one can clearly see from the spreads of the third name that there is a gradual decrease of spreads, in case of survival. When comparing the two figures, one can easily see the effect of introducing intensity correlation. In figure 3 one can see that the only source of correlation is attributed to the jump at the default event. In case of survival the spreads are uncorrelated, although they do have similar drift. In figure 4 one can see the spread correlation. Thus the effect of the copula model is limited to the joint jumps at default times and the downward drift in case of no defaults. This was also argued in Schönbucher and Schubert (2001) and holds for copula functions in general.

In the following figures we show the effect of default and survival information on the CDS spreads of surviving names in case we consider three names out of a basket of 50 names and a basket of 1000 names, respectively. In both cases intensity correlation as well as copula correlation are modelled. The simulated unconditional intensities are the same as in figure 1.

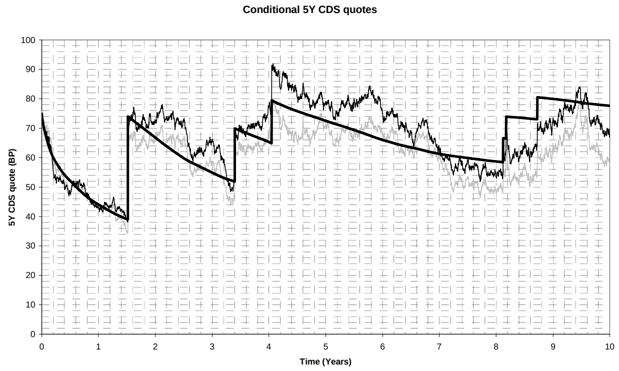

Figure 5: Conditional 5Y CDS spreads for a basket consisting of 50 names.

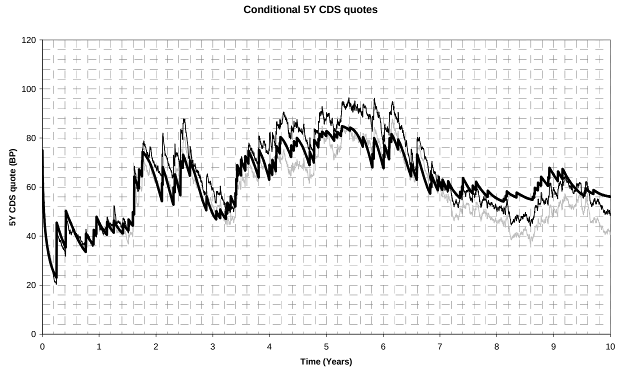

Figure 6: Conditional 5Y CDS spreads for a basket consisting of 1000 names.

From figure 5 one can see that a total of 6 default events have occurred over the 10 year period. Further one can observe that the jump at the first default event was far larger than the jumps at additional default events. This can be explained by the fact that the first default event will give rise to a large change in the information on the common copula factor. Also one can see a very steep decline over the first year, where no default events have occurred. Finally, one can observe the presence of spread correlation in periods of survival.

In figure 6 the number of default events is not easily observed. Over the 10 year period a total of 99 names out of the 1000 have defaulted in this scenario. One can see very erratic behaviour in this graph even for the third name. The third name did not have a random process for the unconditional intensity, so the bevaviour shown in figure 6 can only be attributed to the information changes on the common copula factor. Also note that the decline in spreads in the first few weeks is very steep. The first default event occurs after about 3 months. In this period the CDS spreads have decreased from an initial level of 75 basispoints to as low as 25 basispoints.

When comparing both figures one can note that the effect of default events is larger for the case of the 50 name basket. When a basket of 1000 names is considered, information on the level of the common copula factor is obtained faster.

6 Conclusions

In this paper we have shown how one can price multi-name credit derivatives using the factor copula applied to correlated intensity processes. This means that both default time correlation as well as default intensity correlation, or spread correlation, is modelled. The copula structure generates default time dependency, while the correlated unconditional intensities cause correlation between the spreads. Incorporating both these correlations yields a more realistic model. Using a factor specification it was shown how both these types of correlation can be incorporated into a model while still allowing for fast computation of the multivariate default distribution. One can first condition on the realization of the common factor for the intensity, after which the standard techniques of factor copula models can be applied.

Apart from pricing at initialization, this paper has also shown how one can price conditional on both default events and survival of names in the basket. By translating available information on the individual names to information on the common copula factor we have shown how this can be achieved. Some examples have shown the effect of default events and survival on the CDS spreads of surviving names. It was shown how this effect depends on the timing of the defaults and the number of names in the basket. For larger baskets, information on the realization of the common factor is revealed much faster and default events will have smaller effects than in the case of a small basket. Apart from these results, the simulations have shown that adding intensity correlation to the model will indeed result in more realistic behavior of CDS spreads generated by the model.

In the paper we have focussed on the model in its most simple form, it might also be interesting to allow for more complex behavior, by allowing for correlation between the two common factors and possibly the recovery rate. Thus for instance, our model can allow for a bad state of the economy in which a lot of defaults occur, intensities have shown market-wide increases and recovery rates are low.

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A Intensity Modelling

This appendix gives a more detailed derivation of the distribution of the factors used in the intensities, in case these are of the Vasicek type. As stated in the model description the intensities are modelled by:

$$
\begin{aligned}
d\lambda_i &= \tilde{\rho}_i dh_0 + dh_i \\
dh_j &= \kappa_j \left(\theta_j - h_j\right) dt + \sigma_j dW_j \qquad j = 0, i
\end{aligned}
\tag{27}
$$

In order to determine default probabilities, we are interested in the integral over the intensity, $\int_t^T \lambda_i(s)\, ds$, which follows directly in case we know $\int_t^T h_i(s)\, ds$ for all i. As these h_i terms follow independent Vasicek type processes these integrals have a normal distribution:

$$
\int_t^T h_i(s)\, ds \sim N\left(\mu_i(t,T), \Sigma_i^2(t,T)\right) \tag{28}
$$

In this appendix we will show how the terms $\mu_i(t,T)$ and $\Sigma_i^2(t,T)$ can be derived.

We can easily solve for the terms $h_i(s)$ by integration over the SDE in (27):

$$
h_i(s) = h_i(t)\, e^{-\kappa_i(s-t)} + \theta_i \left(1 - e^{-\kappa_i(s-t)}\right) + \sigma_i e^{-\kappa_i s} \int_t^s e^{\kappa_i u} dW(u) \tag{29}
$$

We next have to integrate the intensity factor over time, this will yield:

$$
\int_t^T h_i(s)\, ds = h_i(t) \int_t^T e^{-\kappa_i(s-t)} ds + \theta_i \int_t^T \left(1 - e^{-\kappa_i(s-t)}\right) ds + \sigma_i \int_t^T e^{-\kappa_i s} \int_t^s e^{\kappa_i u} dW(u)\, ds \tag{30}
$$

The first two terms can be solved easily and this will determine the term $\mu_i(t,T)$:

$$
\begin{aligned}
\mu_i(t,T) &= h_i(t) \int_t^T e^{-\kappa_i(s-t)} ds + \theta_i \int_t^T \left(1 - e^{-\kappa_i(s-t)}\right) \\
&= \frac{h_i(t) - \theta_i}{\kappa_i} \left(1 - e^{-\kappa_i(T-t)}\right) + \theta_i (T - t)
\end{aligned}
\tag{31}
$$

To obtain the last term of (30) we have to switch the order of integration:

$$
\begin{aligned}
& \sigma_i \int_t^T \int_t^s e^{-\kappa_i s} e^{\kappa_i u} dW(u)\, ds \\
= \;& \sigma_i \int_t^T \int_u^T e^{-\kappa_i s} e^{\kappa_i u} ds dW(u) \\
= \;& \sigma_i \int_t^T \frac{1}{\kappa_i} \left(1 - e^{-\kappa_i (T-u)}\right) dW(u)
\end{aligned}
\tag{32}
$$

The integral results in a normally distributed variable, with mean zero. The variance can be determined by applying Ito isometry:

$$
\begin{aligned}
\Sigma_i^2(t,T) &= \sigma_i^2 \int_t^T \frac{1}{\kappa_i^2} \left(1 - e^{-\kappa_i (T-u)}\right)^2 du \\
&= \frac{\sigma_i^2}{\kappa_i^3} \left[\kappa_i (T-t) - 2\left(1 - e^{-\kappa_i (T-t)}\right) + \frac{1}{2}\left(1 - e^{-2\kappa_i (T-t)}\right)\right]
\end{aligned}
\tag{33}
$$

Now it is straightforward to obtain the following expectation:

$$
\mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T h_i(s)\, ds\right)\right] = \exp\left(-\mu_i(t,T) + \frac{1}{2}\Sigma_i^2(t,T)\right) \tag{34}
$$

which is the expectation needed in the paper.