An Implied Loss Model

Martijn van der Voort*

May 11, 2006

Working Paper

Abstract

In this paper we present a model which is, by construction, consistent with observed market quotes for standard CDO tranches. The model is closely related to implied tree methods which can be used for valuing exotic equity derivatives consistent with observed market quotes for vanilla European call and put options. Rather than modelling default events for each name in the basket, the total basket loss is modelled directly and calibrated to CDO prices by construction.

The proposed model has multiple important uses. First, the model can be used as a tool for avoiding arbitrage opportunities when pricing standard CDO tranches. This is a problem which is hard to solve when using the market standard Base Correlation approach in combination with interpolation and extrapolation rules. As a result the proposed model can be used to determine an arbitrage free distribution for portfolio losses for all maturities, which can subsequently be used as input to the more complex HJM type models which have recently become popular. Second, it provides us with a straightforward method for valuing Forward Starting CDOs, FDOs, consistently with observed market quotes on CDO tranches.

A number of tests have been performed which have shown that the model performs well for pricing FDOs, when compared to a number of different factor copula models. Moreover, even under the assumption of heterogeneity of the basket in terms of recovery rates, the performance of the model is still impressive.

Apart from performance tests, some additional tests have been presented in this paper, which show that the limited amount of market data still leads to a large amount of uncertainty in FDO prices.

Finally forward Base Correlation skews implied by the model are considered and these are found to be rather stable.

JEL Classification: G13

Keywords: Credit Risk, Credit Derivatives, CDOs, Forward Starting CDOs, Dependence Modelling, Base Correlation, Factor Copula.

* *Product and Transaction Analysis, ABN AMRO bank and Erasmus University Rotterdam.*
Phone: +31 20 6292941. Mail: <martijn.van.der.voort@nl.abnamro.com>.

The author would like to thank Peter Jäckel, Ioannis Chryssochoos, Sam Phillips and Yunkang Liu for valuable suggestions.

1 Introduction

Over the last years there has been a surge in demand for credit derivatives linked to a basket of reference entities. These developments have led to a liquid market for the standard multi-name credit derivatives, synthetic CDO tranches. Quotes for a number of standard tranches with maturities of 3Y, 5Y, 7Y and 10Y are currently available for the European basket iTraxx and the North American DJ CDX basket. Due to this huge increase in liquidity, market quotes are now used for model calibration and the Base Correlation method, as discussed in McGinty, Beinstein, Ahluwalia, and Watts (2004) Reyfman (2004) and Friend and Rogge (2004), for instance, has become the market standard model for marking CDO tranches to market. This approach is based on the one factor Gaussian copula model with a flat correlation parameter, in combination with an implied correlation skew, or Base Correlation skew.

The next big challenge has been the valuation of more complex multi-name derivatives, the payoff of which depends on the future behaviour of the prices of standard products. Examples are options on CDO tranches, leveraged super senior tranches with loss based triggers, or forward starting CDO tranches. The Factor models currently used for pricing standard CDO tranches are not adequate to value these more exotic credit derivatives. A number of authors have addressed this problem in recent working papers, Andersen, Piterbarg, and Sidenius (2005), Schönbucher (2005) and Bennani (2005). In these works, authors consider a HJM-approach, Heath, Jarrow, and Morton (1992), in order to model the loss distribution of the portfolio over time. An arbitrage free initial distribution for the cumulative loss at future times is taken as input to these models, but no solution is provided to obtain such a distribution, consistent with observed quotes on standard CDO tranches.

In this paper we consider a simple one step Markov-Chain approach as described in Andersen, Piterbarg, and Sidenius (2005) and Schönbucher (2005). However, it is assumed that all transition probabilities are deterministic. Using market data on CDO tranches, one can derive the distribution of the loss for different future times and the forward loss probabilities. By construction the model is calibrated to observed market data. This method is closely related to the implied tree or implied finite difference methods treated in Rubinstein (1994), Derman and Kani (1994), and Andersen and Brotherton-Ratcliffe (1998). Those authors model the stochastic evolution of the stock prices by matching market observed prices for plain vanilla European call and put options. The method discussed here is similar but implies the cumulative loss dynamics for the reference basket, by matching prices of standard CDO tranches. Using this method it is straightforward to enforce the no arbitrage restrictions in the model.

As an application the valuation of Forward Starting CDOs, FDOs, is considered. The performance of the model prices for FDOs is tested using a number of copula models. During such tests it is assumed that we live in a world where all prices are generated by a certain one factor model, for which semi-analytical prices of both CDO tranches and FDOs can be easily derived. Model results can then easily be compared with the 'real' prices in the hypothetical world in order to test the performance of the model. By comparing the proposed method with a multitude of such factor copula models, we are not only able to investigate the performance of the proposed approach, we are also able to assess the influence of the homogeneity assumption[1], as well as the influence of skew interpolation and extrapolation. The interpolation and extrapolation of Base Correlation skews at different maturities has not been given much attention in the literature, but this issue will become more important when moving to more complicated products, while the

[1] *Throughout this paper homogeneity will refer to homogeneity only in terms of the recovery rate. The reference entities can have different marginal default probabilities.*

available number of maturities in the market is still very limited.

The structure of this paper is as follows. The next section gives an overview of well known factor copula models currently used for valuing multi-name credit derivatives. The one factor Gaussian copula is often used and related to this model is the Base Correlation approach. In addition this section discusses a number of alternative factor copula models which have been proposed by various authors in an attempt to better reproduce observed market quotes for standard CDO tranches. Further, FDOs are discussed in this section and the valuation of these products using a factor copula model is explained. The subsequent section focusses on the implied loss model. It shows how prices of CDO tranches can be used in order to determine state probabilities. From state probabilities and some assumptions, the transition probabilities are derived. Furthermore, no-arbitrage conditions are enforced in the model. A comparison of the proposed model with implied tree or local volatility models is given and some applications of the model are discussed. In section 4 the results of some tests are presented which have been used to investigate the performance of the model. In addition, other tests investigate some of the underlying assumptions of the model, such as homogeneity in terms of recovery rates as well as Base Correlation skew interpolation and extrapolation assumptions. Finally, section 5 concludes.

2 Factor Copula Models

The Gaussian copula approach, Li (2000), has become a popular tool for modelling default dependency structures. Using the one factor Gaussian copula allows for semi analytical valuation, which has led to a significant increase in computational speed. Factor Copulas have now been in use for some time and detailed information can be found in e.g. Andersen, Sidenius, and Basu (2003), Laurent and Gregory (2003), or Hull and White (2004).

2.1 Base Correlation

Over the last years correlation modelling has undergone a rapid development, mainly in order to catch up with the developments in the synthetic CDO market. Quotes have become very liquid allowing for implied correlation calculations. With the one factor Gaussian copula approach one can easily calculate implied or compound correlations of market observed CDO tranche quotes. A more appealing approach is known as the Base Correlation method, which was first discussed in McGinty, Beinstein, Ahluwalia, and Watts (2004), Reyfman (2004) and Friend and Rogge (2004), for instance. The main idea of this approach is to rewrite a single tranche as the difference of two equity tranches with different detachment levels. By using the standard one factor Gaussian copula model with flat correlation, market implied correlations for these equity tranches can be determined, which are known as Base Correlations. When plotting these against the corresponding detachment levels the Base Correlation skew appears, which is usually increasing in the detachment level.

The following table presents quotes for tranches on both the iTraxx and the CDX baskets, for January 30^{th}, 2006.

iTraxx	Tranche	Bid	Offer	Delta
5Y, Ref. 36 BP	0 – 3	27.5	28	22.5
	3 – 6	78	80	5.5
	6 – 9	24	26	2
	9 – 12	10	12	1
	12 – 22	5.25	5.75	0.45
7Y, Ref. 48 BP	0 – 3	48	48.5	15
	3 – 6	184	188	8
	6 – 9	46	48	2.5
	9 – 12	26	30	1.5
	12 – 22	12.25	12.5	0.65
10Y, Ref. 58 BP	0 – 3	58	58.5	8
	3 – 6	520	530	11
	6 – 9	98	102	4.25
	9 – 12	42	46	2
	12 – 22	22	24	1

CDX	Tranche	Bid	Offer	Delta
5Y, Ref. 45 BP	0 – 3	35.75	36	19
	3 – 7	107	108	5.5
	7 – 10	26	27.5	1.5
	10 – 15	11	13	0.9
	15 – 30	4.5	6.5	0.35
7Y, Ref. 54 BP	0 – 3	52.75	53.25	11
	3 – 7	238	242	9
	7 – 10	44	48	2.5
	10 – 15	24	26	1.3
	15 – 30	6	8	0.4
10Y, Ref. 67 BP	0 – 3	60.5	60.75	6.5
	3 – 7	630	640	10
	7 – 10	113	117	4.5
	10 – 15	54	57	2.5
	15 – 30	15	17	0.8

Table 1: Quotes of January 30^{th}, 2006 for the standard tranches on both iTraxx and CDX baskets for 5Y, 7Y and 10Y maturities. Quotes for the equity tranches are upfront percentages for trades with 500bp running. All other quotes are in bps.

Here one should note that the quote for the equity tranches is given as an upfront percentage for a trade with 500bps running, which is the market convention for quoting these products.

The resulting Base Correlation skews for iTraxx, with maturities of five years and ten years are given in the following subsection, while those for the CDX index are shown in Appendix B, together with skews implied from alternative models.

Although the Base Correlation approach has some appealing features, it does suffer from some serious drawbacks. It is not clear how to price CDO tranches with non-standard detachment levels and/or non-standard maturities. Interpolation and extrapolation techniques are thus required. For the detachment dimension this is not straightforward, but in the time dimension this is even less clear. Here it is important to note that one can easily obtain tranche prices which allow for arbitrage opportunities when the interpolation and extrapolation methods are not carefully chosen.

In McGinty, Beinstein, Ahluwalia, and Watts (2004) it is proposed to scale the detachment levels down with the expected loss of the corresponding maturity in order to compare Base Correlation skews for different maturities. Intuitively this makes sense as an equity tranche with detachment point of 3% for a period of one year is expected to capture much more of the total cumulative portfolio losses over one year than an equity tranche with 3% detachment considered for a period of ten years. In the remainder of this paper this rule is used for interpolation and extrapolation of the Base Correlation skews. The term scaled detachment point is used to refer to a detachment point, divided by the expected loss for a certain maturity. Here we shall use a simple linear interpolation rule on the observable Base Correlation skews.

Let $\rho_T(x)$ denote the Base Correlation obtained for x, using the Base Correlation skew for maturity T, using the scaled detachment level x. The objective is to determine the correlation $\rho(t, x)$ to be applied to an equity tranche with maturity t and scaled detachment level $x \equiv L/EL_t$, where L denotes the detachment of the tranche and EL_t the expected loss on the portfolio up to maturity t. The following rule is used:

$$\rho(t,x) = \begin{cases} \rho_5(x) & t \leq 5Y \\ \alpha(t,5,7)\cdot\rho_5(x) + (1-\alpha(t,5,7))\cdot\rho_7(x) & 5Y < t \leq 7Y \\ \alpha(t,7,10)\cdot\rho_7(x) + (1-\alpha(t,7,10))\cdot\rho_{10}(x) & 7Y < t \leq 10Y \\ \rho_{10}(x) & t > 10Y \end{cases} \quad (1)$$

$$\alpha(t,T_1,T_2) \equiv \frac{T_2 - t}{T_2 - T_1}$$

In order to calculate the terms $\rho_T(x)$ one needs to resort to interpolation and extrapolation techniques in the (scaled) detachment dimension, as only 5 points are available. In the numerical tests described in section 4 linear interpolation and flat extrapolation is used to achieve this.

2.2 Alternative Factor Copulae

The existence of a Base Correlation skew clearly indicates the failing of the standard Gaussian factor copula to explain market observed quotes on CDO tranches. In an attempt to better explain market quotes, a number of alternative factor copulas have been considered in recent studies. These models often are slight extensions to the well-known one factor Gaussian copula model, but relaxing some of the assumptions. For instance, Andersen and Sidenius (2004) extend the standard model by allowing for random recovery. Another extension proposed by the same authors, is to allow the factor weights to depend on the state of the economy, or the common factor. In this case one can model high correlations when the economy is in a bad state and low correlations when the common factor is high. Hull and White (2004) extend the standard model by modelling both the idiosyncratic terms as well as the common factor by means of the Student t-distribution, which is referred to as the double t copula model. Another simple extension is to model the correlation parameter by means of a discrete distribution, independent of the common factor. Thus the correlation parameter can take on a limited number of values. This model can be easily extended by allowing for random recovery rates. A final extension considered here allows for an external term driving default events. This allows for default events to be caused by an external factor, such as fraud.

A comparative analysis of a number of these and other models is given in Burtschell, Gregory, and Laurent (2005). The following list gives the different models which will be considered in this paper:

- External Defaults Model.
- Random Factor Model.
- Mixture Copula Model.
- Mixture Copula Model with Random Recovery.
- Double t-copula.

More information on these different models can be found in appendix A. In addition, the appendix gives parameter values which have been used for testing. The parameters are chosen such that a reasonable skew for the 5Y iTraxx basket is generated using market data of January 30^{th}, 2006, as shown in table 1. The following figure gives the 5Y Base Correlation skews for iTraxx, resulting from the parameter choices given in the appendix.

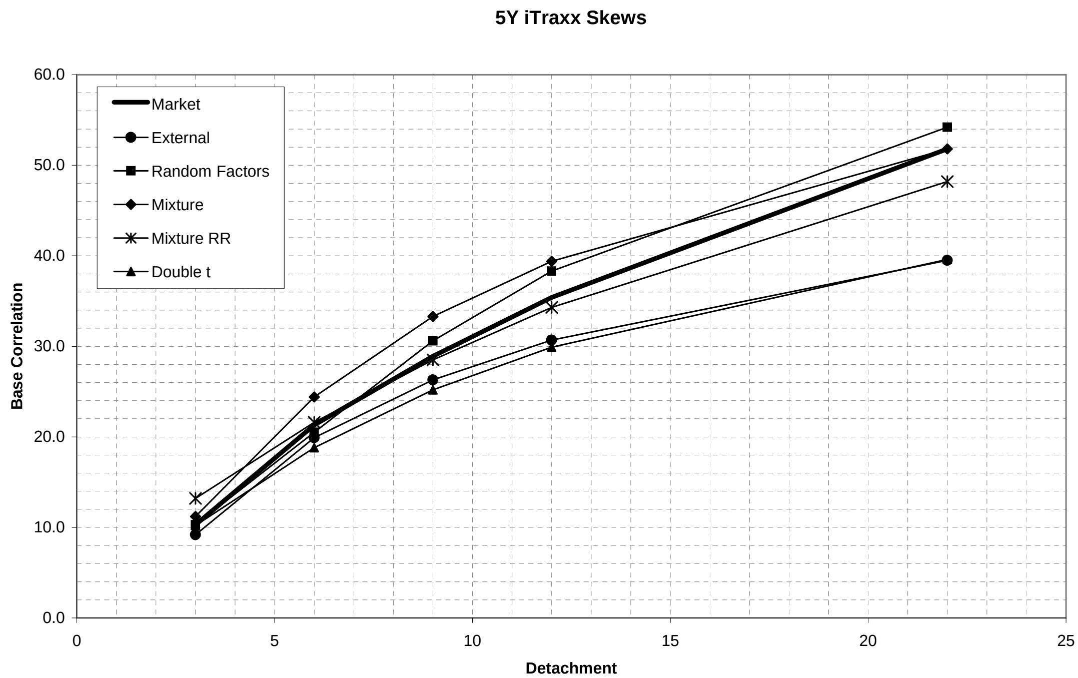

Figure 1: Market and model implied Base Correlation skews for iTraxx 5Y. Different one factor copula models are considered. More details on the models and the parameter choices can be found in the appendix.

From the figure one can observe that the one factor copulas which are considered in this paper are all able to generate upward sloping Base Correlation skews, which resemble the market observed Base Correlation skews. However, choosing parameters to match the 5Y market skew, will often result in a bad fit for the 10Y skew. It appears that all alternatives have difficulties matching the steepness of the 10Y Base Correlation skew implied from market quotes. In the following figure the 10Y skews resulting from the same parameters are given for the different models.

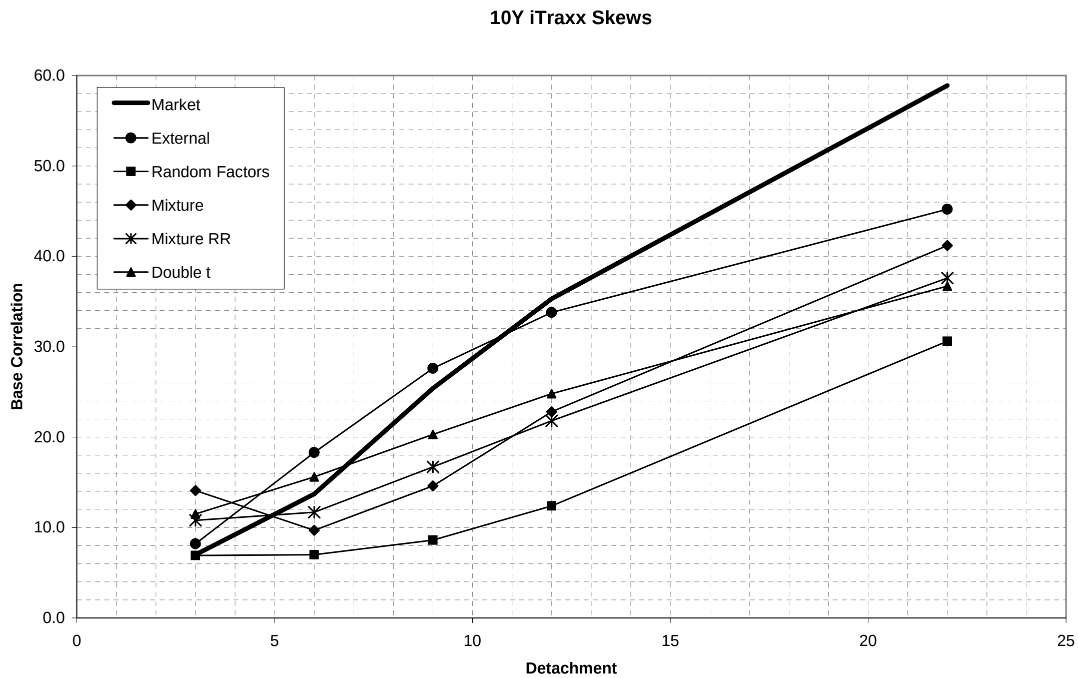

Figure 2: Market and model implied Base Correlation skews for iTraxx 10Y. Different one factor copula models are considered. More details on the models and the parameter choices can be found in the appendix

This figure clearly shows that a poor fit results for most of the models. All models have problems matching the large Base Correlations for the high detachment levels. Further one can observe that the shape of the skew changes for some of the models. For the mixture copula, for instance, one can observe a smile, rather than a skew.

A similar analysis has been performed based on the DJ CDX basket and market data for January 30^{th}, 2006. Plots of the Base Correlation curves, using the same parameter settings, can be found in the appendix.

2.3 Pricing FDOs with Factor Copulas

It is straightforward to price a Forward Starting CDO with each of these factor copulas. One can use the general approach, i.e. first conditioning on the common factor, after which the default events become independent. After conditioning one can determine the probability of a default event for each name falling in the risky period of the FDO. Due to the conditional independence one can easily multiply probabilities. Note that the pricing approach for FDOs is only slightly different from pricing standard tranches, one only has to model default events falling in the risky period. More information on pricing standard CDO tranches with factor copulas can be found in Andersen, Sidenius, and Basu (2003), Laurent and Gregory (2003), or Hull and White (2004).

As was shown above, none of the factor copulas discussed in this paper are able to calibrate exactly to the market. Furthermore, it is not possible to price FDOs using the Base Correlation method. The Base Correlation method should be seen as an advanced interpolation method for CDO prices, but not as a model for defaults. In the following section a model is presented which is able to price FDOs consistently with market observed CDO quotes.

3 The Model

In order to price the FDO consistent with prices for standard CDO tranches, we shall make use of a one step Markov-Chain, or lattice model, to model the process for the cumulative loss of the portfolio over time. The cumulative loss at time t will be denoted by $L(t)$. The modelling approach basically consists of two steps. First a grid is constructed, discretising time and the size of the portfolio loss. For all nodes in this lattice the state probabilities are derived from the observed market data for standard CDO tranches. Next the transition probabilities can be derived under the assumption that only two steps are possible from every node. Either a sideways move, which represents the case of no default over the period, or a single up move, corresponding to a single default event. This lattice model is similar to well-known models in equity derivatives. The derivation of the state probabilities is similar as described in Breeden and Litzenberger (1978) and Dupire (1994). The resulting grid resembles the implied tree approach as discussed in Rubinstein (1994), Derman and Kani (1994), or implied finite difference as discussed in Andersen and Brotherton-Ratcliffe (1998).

In order to price an FDO, one creates the grid only for the risky period of the FDO, that is from the forward starting date T_f up to the maturity, T. We consider the case where the size of the step in the loss direction equals the loss of an average default. Using the Base Correlation method, along with interpolation and extrapolation rules, one can determine the expected loss on tranches, which leads to state probabilities. From the state probabilities one can then determine transition probabilities under the assumption that only one additional loss unit is possible in a single time step.

The following figure gives a graphical representation of the lattice. For simplicity a basket of only 5 names is considered and 7 steps in the time direction have been used.

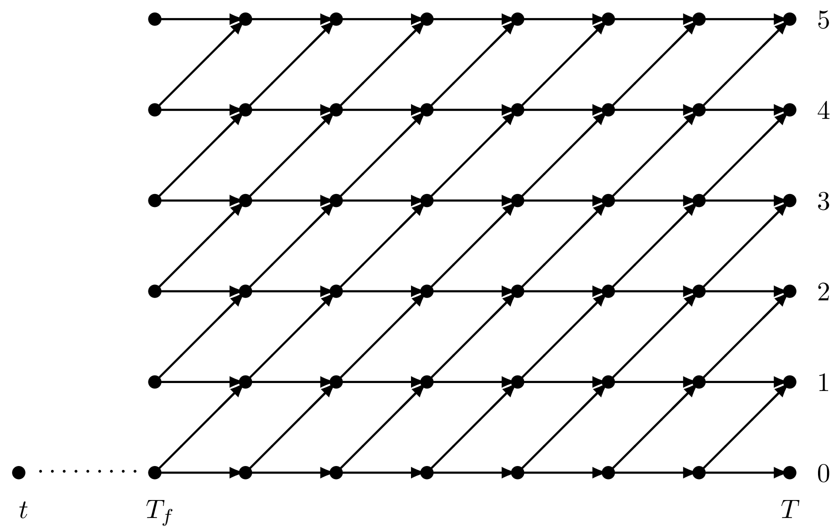

Figure 3: A lattice approach which is used for modelling the cumulative loss process. The grid discretises both the time domain and the losses domain. It is assumed that one step upwards corresponds to a single default.

At the right end of the figure, the number of defaults are listed for each height in the lattice. Throughout the remainder of this paper the indices i will be used for the time step. Indices j and k will be used for the height in the grid, i.e. the number of loss units. Further we shall use the notation ΔL for the loss unit and Δt for the time unit.

3.1 State Probabilities

The first step of the modelling approach is to determine the probability of being at a certain node at a certain time. This is known as the state probability and we use the notation $\delta_{i,k}$ to denote the probability of observing a loss of $k \cdot \Delta L$ at time t_i:

$$\delta_{i,k} \equiv Pr\left(L(t_i) = k \cdot \Delta L\right) \tag{2}$$

In order to determine these state probabilities the expected loss of equity CDO tranches is considered. In order to calibrate the model to the market, this expected loss can be determined using the Base Correlation approach. Let $E_{i,j}$ denote the expected loss on an equity tranche with maturity t_i and detachment level $j \cdot \Delta L$.

One can easily determine the expected loss in the model, under the assumption that the cumulative losses can only be at one of the nodes in the grid. This model value must match the value obtained from market data, using the Base Correlation approach:

$$E_{i,j} = \Delta L \sum_{k=0}^{j} k \cdot \delta_{i,k} + j \cdot \Delta L \sum_{k=j+1}^{n} \delta_{i,k} \quad (3)$$

Here n gives the total number of names in the basket and thus the maximum number of loss units. Now from (3) it follows that it is fruitful to consider the difference in expected loss, for two consecutive equity tranches:

$$\frac{E_{i,j+1} - E_{i,j}}{\Delta L} = \sum_{k=j+1}^{n} \delta_{i,k} \quad (4)$$

Thus the difference directly gives us information about the cumulative loss distribution at time $i \cdot \Delta t$. One additional differencing will then result in the transition probabilities:

$$\delta_{i,j} = \begin{cases} \frac{2E_{i,j} - E_{i,j+1} - E_{i,j-1}}{\Delta L} & j > 0 \\ 1 - \frac{E_{i,1}}{\Delta L} & j = 0 \end{cases} . \quad (5)$$

Here one should note the following trivial cases: $E_{i,n+1} = E_{i,n}$ and $E_{i,0} = 0$.

The analysis above shows how one can determine state probabilities using the expected loss of tranches in a consistent way with observed marked prices. The analysis is closely related to deriving implied distributions from European call and put options in the equity market. An equity tranche can be seen as a long position in the portfolio loss, combined with a short call option on the loss, with strike equal to the detachment point. The results above are thus not surprising, taking derivative with respect to strike, or detachment point, leads to the cumulative distribution function for the basket loss. Taking second derivatives thus gives the probability density function.

3.2 Transition Probabilities

Once all the state probabilities have been determined at the different times, the next step is to determine the transition probabilities. These are defined to be the probabilities of moving from a certain state at time t_i to a certain state at t_{i+1}. Here one important observation is the fact that cumulative losses will never decrease over time. In the grid this means that only sideways or upward steps can be taken. The transition probabilities satisfy two restrictions, which are listed below:

- The state probabilities should be matched.
- Transition probabilities originating from one point must sum to 1.

These two restrictions leave room for some freedom. In order to fill these degrees of freedom, one can use the following additional restriction:

From a certain node in the grid to the next time step, the loss can either stay at the same level, or increase by one single loss unit

At first sight, this seems to be a restrictive assumption and the effect of this assumption is not directly clear as it relates to the discretisation sizes in both directions. When the time step is large, allowing only for an increase of the loss of one unit is rather restrictive. Especially when the size of the loss unit is taken to be small. However when one chooses the grid spacing in the loss direction to be equal to the size of an average loss on default, the assumptions thus allows for at most one single default over the time period. Making finer partitions in the loss dimension is not desirable as the loss distribution is not continuous, but can be seen as a jump process.

Let $p_{i,j,k}$ denote the probability of having a loss of $k \cdot \Delta L$ at time t_{i+1}, given that the cumulative portfolio loss equals $j \cdot \Delta L$ at time t_i :

$$p_{i,j,k} \equiv Pr\left(L(t_{i+1}) = k \cdot \Delta L \mid L(t_i) = j \cdot \Delta L\right) \tag{6}$$

Using the two restrictions plus the additional assumption as discussed above, it is straightforward to derive the transition probabilities. The assumption of at most one additional loss unit will result in a lot of zero transition probabilities:

$$p_{i,j,k} = \begin{cases} \frac{\delta_{i+1,k} - \delta_{i,k-1} \cdot p_{i,j-1,k}}{\delta_{i,j}} & if \quad j = k \\ 1 - p_{i,j,k-1} & if \quad j = k-1 \\ 0 & else \end{cases} \tag{7}$$

Here, for notational convenience, one should set all values with negative indices equal to zero. This results in an algorithm which moves from left to right and from low to high. The first line makes sure that the transition probabilities lead to the right state probabilities, by taking into account the amount contributed by the lower node. The second line assures that the transition probabilities for moves originating from one node will sum to 1.

One should make adjustments in case the transition probabilities fall outside the unit interval, in order to avoid arbitrage opportunities in the grid. In order to get the best fit, it is recommended to start determining the transition probabilities for the first time step and move to the maturity. After one has made adjustments to avoid wrong probabilities, one should recalculate the realised state probabilities, using the adjusted probabilities. These should then be used while determining the transition probabilities to the next step.

3.3 Dealing with Arbitrage

From the analysis above, it appears that one can first determine all state probabilities and subsequently derive the transition probabilities from these. However, implied tree methods as outlined in Rubinstein (1994), Derman and Kani (1994) are recursive algorithms, which start at the valuation date and move recursively to the desired maturity by determining at each step transition probabilities followed by state probabilities. Theoretically these two approaches should result in the same state and transition probabilities. This will be the case when the CDO prices used as input to the algorithm are arbitrage free. However, adjustments are required when this is not the case, in order to avoid negative state or transition probabilities.

The best way to deal with such situations is by constructing the grid recursively, starting at the forward starting date and moving to the maturity of the FDO. State probabilities at a fixed time step can easily be made arbitrage free, by making sure the expected loss on equity tranches is non-decreasing in the detachment level. In addition the changes in expected loss should be non-decreasing. When determining transition probabilities, going from one time step to the next, one must make sure that these transition probabilities, calculated from (7) are within the $[0, 1]$

interval and adjust them if necessary. If adjustments have been made at a certain time step, one can recalculate the actual state probabilities at the end of the time step, using the adjusted transition probabilities. One can then continue the algorithm by moving to the next time step. When the recalculation of the state probabilities is not performed, errors might accumulate over time, leading to a poor fit of the tranche prices used as input.

As discussed there is no need for adjustments when input prices are arbitrage free. One should keep in mind however that in practice only a limited number of quotes are available, with 3Y, 5Y, 7Y and 10Y tenors. Interpolation and extrapolation is thus required both in the detachment domain as well as the maturity domain. Using simple interpolation and extrapolation rules is likely to lead to arbitrage opportunities in tranche prices.

3.4 Comparison with Implied Volatility Trees

The method described above is very similar to the implied tree methods discussed in Rubinstein (1994), Derman and Kani (1994). The drawbacks of the use of such models are well known. For instance, the forward smiles implied by the model do not show realistic behavior.

As we have introduced a similar method for modelling the dynamics of the cumulative loss on a basket of reference entities, one can expect similar problems. However the situation for loss modelling is different in multiple ways. First, in contrast to implied trees for equity derivatives, the cumulative loss on a portfolio is by definition non-decreasing over time. Moreover it is reasonable to assume that there can be only one default in a small time step, especially when a small time step is used in the construction of the tree.

Second the no-arbitrage conditions are totally different. For the equity implied trees, one has to ascertain that the forward restriction is satisfied over each time step. Thus the restriction is dynamic and needs to hold for every height in the tree and at every time step. For the loss tree the situation is different, as the restriction on the expected loss is automatically taken into account by matching the tranche prices. The forward restriction is not dynamic in this case.

It does suffer from the fact that the model assumes that future transition probabilities are deterministic, which for implied trees known from equity derivatives, leads to unrealistic future volatility smiles, i.e. Andersen and Andreasen (2000). However the clear differences between both cases does not allow immediate conclusions on whether the method is appropriate or not. In the following section we therefore present some tests in order to asses the consequences of the proposed approach.

3.5 Applications

The grid or lattice approach discussed in this paper has multiple uses. In this paper the main focuss will be on FDO pricing. However, the model is also capable of pricing more complex multi-name credit derivatives, such as leveraged super senior tranches or options on tranches. However, in the approach discussed here only the cumulative portfolio loss is modelled and not the randomness in default probabilities, or randomness in spreads. This latter randomness is of importance when pricing these more complex multi-name credit derivatives and thus the model discussed here can not be considered adequate for these purposes.

Here we discuss two applications for which the model can be useful. As a first application, it is shown how one can use the model to price Forward Staring CDO tranches. Second it is shown how the model can be used to price standard CDO tranches as an improvement of the Base Correlation pricing method, by avoiding arbitrage opportunities.

3.5.1 FDO Pricing

Once the grid has been created FDOs can be easily priced by first conditioning on the state of the portfolio at the forward starting date, T_f. From the state probabilities one can directly obtain the probability of reaching a certain loss level at the forward date: $Pr(L(T_f) = k \cdot \Delta L)$ for all k. Next one conditions on the event $L(T_f) = k$. Let e_k denote the $k-th$ unit vector, i.e. a vector with all zeroes except at element k, which is set to 1. Using the transition probabilities one can propagate this vector, creating conditional state probabilities up to the maturity of the Forward Starting CDO, T. With these conditional state probabilities one can easily determine the expected loss on the tranche at every time step in the risky period by considering the additional default events. One should then multiply with the conditioning probability: $P(L(T_f) = k)$ and sum these numbers for all possible future states of the basket.[2] Finally one can easily price the FDO using standard techniques for CDOs. Results of such calculations are presented in the next section, where a comparison is made between the proposed model and the one factor copula models discussed in subsection 2.2.

3.5.2 Arbitrage Free Pricing of CDOs

The proposed model is also useful to obtain arbitrage free prices for synthetic CDOs.

The Base Correlation approach suffers from interpolation and extrapolation problems, both in the time and detachment dimensions. In the market only tranche quotes are available for 5 different detachment levels and for 3 or 4 maturities.[3] In order to price a CDO with non-standard attachment / detachment levels, or with a non-standard maturity interpolation and extrapolation of the different skews is required. It is not guaranteed that this will lead to arbitrage free prices. A related problem is the pricing of tranches on bespoke baskets. Again one wants to use the information available in the market quotes, but arbitrage opportunities can easily arise.

In order to avoid these problems, the model discussed in this paper can be used. One can choose a certain interpolation and extrapolation method for the Base Correlation skews derived from market quotes and apply these rules to obtain standard CDO prices required as input to the model. During the construction of the grid, arbitrage can be avoided by following the instructions as outlined in subsection 3.3. Obviously the quotes used as input to the model will not always be matched, but for reasonable choices for interpolation and extrapolation, one can expect that the model will still match the market observed quotes at the liquid maturities and in addition is arbitrage free.

This application will also be very useful when modelling with the more advanced forward loss modelling frameworks as described in Andersen, Piterbarg, and Sidenius (2005) and Schönbucher (2005), for instance. These frameworks assume an arbitrage free distribution at time zero is given as input. It is not straightforward to obtain such a full arbitrage free distribution both in the time domain as well as the detachment domain, given only a very limited amount of market quotes. The approach outlined in this paper can provide such a distribution.

[2] *In order to speed up the algorithm one can choose to ignore the conditioning states which have low state probabilities. This will also allow for truncation of the grid.*

[3] *Quotes are usually available for 5Y, 7Y and 10Y maturities. Less liquid are tranches with a 3Y maturity.*

4 Numerical Tests

In this section the results of a number of tests are presented. Four aspects of the implied loss model are tested. First the FDO prices resulting from the implied lattice approach are compared with one factor copula models. Here it is important to note that the other models are static default models.

Second the effect of the homogeneity assumption, in terms of recovery rates, is investigated. Again a comparison is made using alternative factor copula models, where recovery rates are heterogenous.

Third, the influence of skew interpolation and extrapolation on the price of the FDO is investigated. Again, alternative factor copula models are used.

Finally the model is applied to market data for CDO tranches. Using the Base Correlation method and simple interpolation and extrapolation rules the prices of FDOs are determined. These results are subsequently used to derive forward Base Correlations.

4.1 Data

We consider a number of forward starting FDOs, differing in both subordination and tranche width, as well as maturity and forward starting date. All FDOs are referencing either the iTraxx or CDX basket, which consist of 125 high grade European and North American names, respectively. For all 250 reference entities entire term structures of CDS quotes are available of January 30^{th}, 2006. These CDS quotes are used to calibrate marginal default distributions for every single name in the basket. Furthermore we take the swap market quotes for that day to derive a proxy for the risk free discount factors. Finally, the CDO data as presented in table 1 has been used, for some of the tests.

For calculation purposes a lattice has been created using spacings in the time dimension of two weeks. In the loss direction we use a step size equal to the average loss of one single default event. In most test a recovery rate of 40% has been used, but this assumption is relaxed when testing the heterogeneity assumption.

4.2 Model Performance

In this subsection the quality of the model is assessed. To achieve this it is assumed that we live in a world were all prices are generated by one of the models mentioned in subsection 2.2. Parameter settings for these models can be found in the appendix. As was shown, most of these models are able to generate reasonable Base Correlation skews and all them can easily be used to price FDOs without the need of additional assumptions. These values are compared to the ones generated by the proposed model. As discussed, the proposed model requires the expected loss of tranches for all detachment levels and for different maturities and the alternative factor copula models are used to generate these. It is important to note that this can be done in different ways. First one can directly use the factor copula under consideration to generate the expected loss on a certain tranche. Alternatively, one can generate 5Y and 10Y Base Correlation skews from the alternative models and use these as input, along with an interpolation and extrapolation rule. Both these approaches are considered. The first approach is used in this subsection in order to test the performance of the model, while the latter is used in the following subsection to investigate the influence of interpolation and extrapolation of Base Correlation skews on prices of FDOs.

4.2.1 Homogenous Basket

As a first test a homogenous basket is considered where all names are assumed to have a recovery of 40%. Thus a grid is constructed using a loss unit of 60% divided by the 125, the number of names in the basket. The grid is constructed using time steps of two weeks, commencing from the forward starting date. The maturity is also included in the grid, thus making the final step size somewhat smaller.

The following table lists the fair premia of the deals under the five alternative models. For each alternative model two rows of results are presented. The first row displays the exact result according to the model. Every second row presents the corresponding results using the proposed implied lattice approach. The lattice is constructed using CDO prices as generated by the alternative copula model under consideration.

Model	Pricing	0% – 3%	3% – 6%	6% – 9%	9% – 12%	12% – 22%
External Defs.	Exact	1346	194	76	49	26
	Lattice	1364	194	73	47	25
Mixture Copula	Exact	1413	249	38	28	18
	Lattice	1419	252	37	26	17
Mixt. Copula RR	Exact	1383	264	66	27	12
	Lattice	1385	266	65	27	11
Rand. Fact. Loadings	Exact	1502	300	35	7	5
	Lattice	1498	302	36	7	6
Double t Copula	Exact	1330	260	75	38	17
	Lattice	1346	257	72	36	17

Table 2: Investigating the errors made when assuming actual market prices are generated by an alternative model. Five alternative models are considered and Fair premia are given in BP.

From the table one can observe that the results of the implied loss model are close to those of the alternative one factor copula models. The performance of the model against the external defaults model and the double t-copula model is least impressive. The performance can be increased by decreasing the step in the time direction, however one can still expect minor differences. In the following table results are given for a 1Y forward CDO with maturity of 4Y following the forward date.

Model	Pricing	0% – 3%	3% – 6%	6% – 9%	9% – 12%	12% – 22%
External Defs.	Exact	1245	84	34	20	10
	Lattice	1251	83	33	20	9
Mixture Copula	Exact	1219	71	28	21	13
	Lattice	1223	71	27	21	13
Mixt. Copula RR	Exact	1195	118	31	16	8
	Lattice	1197	118	31	15	7
Rand. Fact. Loadings	Exact	1274	91	11	9	8
	Lattice	1273	92	11	9	9
Double t Copula	Exact	1229	105	30	17	8
	Lattice	1236	103	30	17	8

Table 3: Investigating the errors made when assuming actual market prices are generated by an alternative model. Five alternative models are considered and Fair premia are given in BP.

From the table one can observe that the results of the implied loss model are close to those of the alternative one factor copula models. The performance is worst for the external defaults model and the double t-copula.

It is interesting to see from both table 2 and table 3 that fair premia can be quite different for the various alternative models, but the proposed model always results in a close fit.

4.2.2 Heterogenous Basket

The approach discussed in this paper models the portfolio loss directly and thus makes no distinction between the different names in the basket. The grid is constructed such that each step in the loss dimension corresponds to the loss of an average default event. This seems to be a restrictive assumption thus some tests are performed in order to investigate this.

Instead of the 40% recovery for every name in the basket it is assumed that the 125 names have different recovery rates. The 125 names in the iTraxx basket are split into 25 groups of 5 names. For group i the recovery rate, R_i, is set to: $R_i = 10\% + i \cdot 2.5\%$, for $i = 0, \ldots, 24$. Thus recovery rates are spread between 10% and 70%. The assignment to the different groups is done based on the 5Y CDS quote for each name such that names with smallest CDS spreads have smallest recovery. The same procedure is applied to the names in the CDX basket. In the grid we choose the loss unit equal to the average recovery rate, which under these settings again equals 40%.

Model	Pricing	0% – 3%	3% – 6%	6% – 9%	9% – 12%	12% – 22%
External Defs.	Exact	1350	205	79	46	19
	Lattice	1364	205	76	44	19
Mixture Copula	Exact	1451	219	36	26	15
	Lattice	1448	226	35	25	15
Mixt. Copula RR	Exact	1306	258	82	34	14
	Lattice	1305	261	82	34	12
Rand. Fact. Loadings	Exact	1524	263	24	7	5
	Lattice	1512	268	26	7	5
Double t Copula	Exact	1357	254	69	32	13
	Lattice	1365	253	67	31	13

Table 4: Investigating the errors made when assuming actual market prices are generated by an alternative model. Five alternative models are considered and Fair premia are given in BP. Recovery rates are between 10% and 70% and names with smallest CDS spreads have smallest recovery rates.

From the results in table 4, one can observe that differences between the correct results and those generated by the implied grid method are again reasonably close. One can not clearly observe larger differences than those in table 2. Thus it appears that the homogeneity assumption in terms of the recovery rates does not lead to large errors in pricing when using the implied grid approach.

4.3 Skew Interpolation and Extrapolation

The above tests show that the method leads to accurate FDO values when using the correct CDO values to construct the grid. During testing the state probabilities of the grid were directly determined using one of the factor copulas discussed in section 2. In practice, the situation is less ideal as one can observe only 5 tranches per index for a limited number of maturities, usually 3Y, 5Y, 7Y and 10Y, where the 5Y quotes are most liquid. For a 3Y maturity, usually only the equity tranche is quoted. This limited number of maturities means that one has to resort to interpolation and extrapolation techniques to obtain input CDO prices required for the algorithm. Interpolation and extrapolation is required in both the loss dimension as well as the time dimension. In order to test the influence of the interpolation on the quality of the results generated by the model similar tests will be performed as discussed above, but the input is generated in a different way. The alternative models are used to create tranche quotes for both 5Y and 10Y maturities. These are then used as input in the Base Correlation method to generate Base Correlation skews for 5Y and 10Y maturities. Finally the grid method is applied using the Base Correlation method to price all required tranches. The final step allows for a large amount of freedom, as there are multiple ways to price non-standard tranches and/or tranches with non-standard maturity given a skew. In this paper a method is used based on the scaled detachment idea proposed in McGinty, Beinstein, Ahluwalia, and Watts (2004), see formula (1).

Below a summary of the approach is given, which will be used in order to investigate the effect of limited market data on the potential error in prices of FDOs.

- Choose a one factor copula
- Determine fair premia for standard tranches, for 5Y, 7Y and 10Y maturities.
- Use these as input to generate Base Correlation Skews.
- Use these skews and interpolation and extrapolation using (1) to determine all CDO prices required by the model.

Using this approach will lead to errors in prices due to two reasons. First, the lack of required market data will inevitably lead to errors in pricing. Second, the quotes in the market are given as a fair premium and the one factor Gaussian model is required to translate these to expected loss and thus eventually to the implied loss distribution.

Model	Pricing	0% – 3%	3% – 6%	6% – 9%	9% – 12%	12% – 22%
External Defs.	Exact	1346	193	76	49	26
	Lat. 5,7	1445	219	66	40	19
	Lat. 5,10	1443	235	69	37	13
Mixture Copula	Exact	1413	249	38	28	18
	Lat. 5,7	1434	303	51	19	13
	Lat. 5,10	1457	280	39	21	9
Mixt. Copula RR	Exact	1383	264	66	28	12
	Lat. 5,7	1402	290	72	28	10
	Lat. 5,10	1401	293	70	27	8
Rand. Fact. Loadings	Exact	1502	300	35	7	5
	Lat. 5,7	1468	315	46	6	2
	Lat. 5,10	1496	296	51	13	2
Double t Copula	Exact	1330	260	75	38	17
	Lat. 5,7	1402	281	73	31	12
	Lat. 5,10	1419	284	72	29	8

Table 5: Investigating the errors made when assuming actual market prices are generated by an alternative model, for tranches, with 5Y, 7Y maturities or 5Y, 10Y maturities. Subsequently these are used as input to the Base Correlation approach and using scaled detachment point interpolation / extrapolation gives the required input to the Implied Grid approach. Five alternative models are considered and Fair premia are given in BP.

From the results it can be observed that the differences have become much larger than before.

It is important to note that the large differences do not indicate a weakness of the proposed model. It merely gives an indication of the potential error in FDO prices due to lack of market data and due to the model assumption required to translate fair premia to expected loss.

4.4 Forward Base Correlation Skews

As discussed in Andersen and Andreasen (2000), local volatility models used for equity derivatives pricing can lead to unrealistic future implied volatility skews. Moreover, these models often predict that the skew will vanish in the future. As discussed in subsection 3.4, there are some significant differences when comparing the implied loss model with local volatility models. It will thus be very interesting to investigate the future Base Correlation skew, as predicted by the implied loss model. In order to determine such forward Base Correlations, we apply the one factor Gaussian copula model to forward default probabilities as seen from today. I.e. let τ_i denote the default time for reference entity i. Further let $\mathcal{F}_t$ denote the filtration generated by loss process. For the forward skew at time T_f, as seen from time t we use the probabilities $Pr\left(T_f < \tau_i \leq T \left| \mathcal{F}_t \right.\right)$ as input to the one factor Gaussian copula. This will then lead to prices of FDOs in a similar manner as one would price standard CDO tranches and the Base Correlation method can be applied using the lattice results for FDO prices as input.

In the tests the actual market data of January 31, 2006, for CDS quotes, CDO quotes and interest rate swap quotes has been used. Base Correlation skews are determined and interpolation and extrapolation rules as discussed in subsection 2.1 have been used. Prices for Forward Staring CDOs are computed using the implied loss model, where the time steps of 2 weeks have been used. Standard iTraxx detachment levels are considered and the maturity of the FDOs is set 5Y after the forward starting date. Forward starting periods of 1Y, 3Y and 5Y are considered and this allows for investigating the 1Y, 3Y and 5Y forward Base Correlation skew for a 5Y maturity. The following figure presents the results of this exercise.

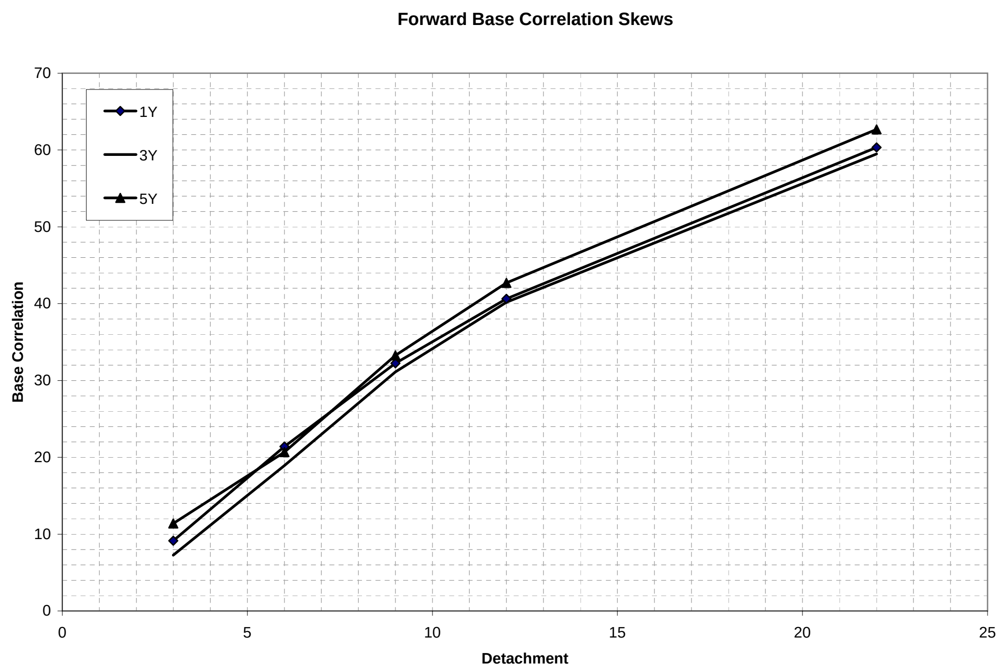

Figure 4: Forward Base Correlation Skews as implied by the lattice model. The Base Correlation approach and simple interpolation and extrapolation rules are used to generate required input for the model. Forward tenors are set to 1Y, 3Y and 5Y and the skew tenor is set at 5Y.

It is interesting to observe that the forward skews do not differ much. Moreover they look similar as the the instantaneous Base Correlation curve as shown in figure 1, but are somewhat steeper as the forward Base Correlations skews end at around 60%, while the instantaneous Base Correlation skew, as shown in figure 1, ends at a level of about 52%.

5 Conclusions

Over the last years the market for credit derivatives has undergone some mayor developments. The market for CDO tranches has become very liquid, to such an extend that modelling can be done on a marking to market basis.

In this paper a simple implied loss model has been presented. This is a lattice approach which is by construction consistent with observed market quotes for CDO prices. This implied loss model has multiple important uses. First, it provides an easy way to obtain arbitrage free prices for CDO tranches. This is a mayor improvement compared to using the Base Correlation method in combination with interpolation and extrapolation rules. Moreover, an arbitrage free loss distribution is obtained which can be used in more complicated HJM type models, discussed in recent literature in Schönbucher (2005) and Andersen, Piterbarg, and Sidenius (2005), for instance. As a second application we have considered the pricing of forward starting CDO tranches, FDOs. Using a number of popular one factor copula models, most of which are able to generate reasonable Base Correlation skews, we were able to test the performance of the implied loss model. It was shown that the proposed method gives accurate results and that the homogeneity assumption, in terms of recovery rates, does not lead to large pricing errors. In addition the effect of skew interpolation and extrapolation in the time domain was investigated. It was shown that the limited amount of available market data still leads to a large amount of uncertainty in FDO prices. Finally some tests were performed in order to investigate the model implied forward Base Correlation skews. It was found that these were rather stable and not very different from the instantaneous Base Correlation skews as observed in the market.

References

Andersen, L. and J. Andreasen (2000). Jump-diffusion processes: Volatility smile fitting and numerical methods for option pricing. *Review of Derivatives Research 4*, 231–262.

Andersen, L. and R. Brotherton-Ratcliffe (1998). The equity option volatility smile: A finite difference approach. *Journal of Computational Finance 1*(2), 5–38.

Andersen, L., V. Piterbarg, and J. Sidenius (2005, October). A new framework for dynamic credit portfolio loss modeling. Working paper.

Andersen, L. and J. Sidenius (2004, Winter). Extensions to the gaussian copula: Random recovery and random factor loadings. *The Journal of Credit Risk 1*(1), 29–70.

Andersen, L., J. Sidenius, and S. Basu (2003). All your hedges in one basket. *RISK November*, 67–72.

Bennani, N. (2005, November). The forward loss model: A dynamic term structure approach for the pricing of portfolio credit derivatives. Working paper, The Royal Bank of Scotland.

Breeden, D. and R. Litzenberger (1978). Prices of state-contingent claims implicit in options prices. *Journal of Business 51*, 621–651.

Burtschell, X., J. Gregory, and J.-P. Laurent (2005, April). A comparative analysis of CDO pricing models. Working paper.

Derman, E. and I. Kani (1994, February). Riding on a smile. *RISK 7*, 32–39.

Dupire, B. (1994, January). Pricing with a smile. *RISK 7*, 18–20.

Friend, A. and E. Rogge (2004). Correlation at first sight. *Forthcoming: Economic Notes: Review of Banking, Finance and Monetary Economics*.

Heath, D., R. Jarrow, and A. Morton (1992). Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation. *Econometrica 60*, 77–105.

Hull, J. and A. White (2004). Valuation of a CDO and an n-th to default CDS without monte carlo simulation. *Journal of Derivatives 2*, 8–23.

Laurent, J.-P. and J. Gregory (2003). Basket default swaps, CDO's and factor copulas. Technical report, BNP Paribas and ISFA Actuarial School, University of Lyon.

Li, D. (2000). On default correlation: A copula approach. *Journal of Fixed Income 9*, 43–54.

McGinty, L., E. Beinstein, R. Ahluwalia, and M. Watts (2004, March). Introducing base correlations. *Credit Derivatives Strategy, JP Morgan*.

Reyfman, A. (2004, May). *Valuing and Hedging Synthetic CDO Tranches Using Base Correlations*. Bear Stearns & Co. Inc.

Rubinstein, M. (1994, July). Implied binomial trees. *Journal of Finance 49*(3), 771–818.

Schönbucher, P. (2005, September). Portfolio losses and the term structure of loss transition rates: A new methodology for the pricing of portfolio credit derivatives. Working paper, ETH Zürich.

Voort, van der, M. (2005). Factor copulas: Totally external defaults. Working paper, ABN AMRO Bank and Erasmus University Rotterdam.

A Factor Copula Models

Applying the standard one factor copula model to observed market data gives rise to a Base Correlation skew, McGinty, Beinstein, Ahluwalia, and Watts (2004). A number of alternative models have been proposed in the literature and this appendix gives a brief discussion of some of these alternatives.

A.1 Standard One Factor Gaussian Copula Model

The Gaussian copula model, Li (2000) has become the standard for modelling default dependency structures. Imposing the restriction that correlation is driven by a single factor allows for semi-analytical pricing of the liquid multi-name products as described in Laurent and Gregory (2003), or Hull and White (2004).

All models discussed in this paper are closely related to the standard one factor Gaussian copula. Defaults are generated by a latent variable for each name. Correlation is modelled by correlating the latent variables in some specific way. For the standard model with flat correlation, the latent variables are modelled as follows:

$$\begin{aligned} X_i &\equiv \rho \cdot Y + \sqrt{1-\rho^2} \cdot \xi_i \\ \tau_i \leq t &\iff X_i \leq \chi_i(t) \end{aligned} \tag{A.1}$$

Here X_i denotes the latent variable and a default event has occurred before time t, when it is smaller than $\chi_i(t)$. The latent variables of the different names are correlated through the common factor, Y, which is assumed to have a standard normal distribution. The second term depends on the idiosyncratic random variable ξ_i, which also has a standard normal distribution. All these random normals, i.e. Y and ξ_i are independent. It is easy to see that the latent variables X_i have a standard normal distribution as well. The distribution of these latent variables determine the thresholds $\chi_i(t)$:

$$\chi_i(t) = F_{X_i}^{-1}(p_i(t)) \tag{A.2}$$

Here $p_i(t)$ is the marginal default probability for name i for the period up to time t. These are obtained by bootstrapping to CDS quotes from the market. Further the function F_{X_i} denotes the cumulative density function for the latent variable. Here this is the standard normal distribution, but this differs in the alternatives we will consider.

In the following subsections some alternatives to the standard model are discussed. These alternatives are all simple extensions to (A.1).

A.2 Mixture Copula

One of the simplest extensions to the standard one factor Gaussian copula is the mixture copula. Instead of a fixed correlation, one assumes that the copula correlation can take on a finite number of values. Using just two or three possible outcomes will already result in reasonable Base Correlation skew. The following formula describes this copula:

$$\begin{aligned} X_i &\equiv \rho \cdot Y + \sqrt{1-\rho^2} \cdot \xi_i \\ \rho &= \alpha_j, \quad \textit{with probability } p_j, \quad \textit{for } j = 1, \ldots, M \end{aligned} \tag{A.3}$$

Here, the common factor is denoted by Y, the latent variables by X_i. The term ρ is the factor loading and determines the correlation. Here it is random and can take on M different values. Note that the realisation of the correlation parameter is independent of the common factor, this in contrast to the random factor loading model of Andersen and Sidenius (2004), discussed below.

Calculations with this model are straightforward. One first conditions on the realisation of the correlation parameter, after which one can apply the techniques known from the standard framework. This is to be repeated until all M possible realisations for the correlation parameter have been considered. Finally one can take the probability weighted sum in order to determine the value of both legs of the deal, and thus the fair premium.

In the paper we consider a two state mixture copula with the parameters as listed in the table below:

Parameter	State 1	State 2
Probability	80%	20%
Correlation	0%	80%

Thus the economy can be in two states. There is a probability of 80% that it will be in a state where names are uncorrelated and a probability of 20% that there is a large correlation of 80%.

A.3 Mixture Copula with Random Recovery

The mixture copula discussed above can easily be extended to allow for random recoveries. One can assume that apart from different states for correlation the recovery rates can be in different states as well. For instance one can use this to model a bad state of the economy where high correlations go accompanied by small recovery rates. The model is a simple extension to (A.3):

$$\begin{aligned} X_i &\equiv \rho \cdot Y + \sqrt{1-\rho^2} \cdot \xi_i \\ \rho &= \alpha_j, \quad \textit{with probability } p_j, \quad \textit{for } j = 1, \ldots, M \\ R_i &= \beta_j \cdot \mathbb{E}\left(R_i\right), \quad \textit{with probability } p_j, \quad \textit{for } j = 1, \ldots, M \\ \sum_{j=1}^{M} p_j \cdot \beta_j &= 1. \end{aligned} \tag{A.4}$$

The interpretation of this model is similar as in subsection A.2. R_i denotes the recovery rate for name i and this equals the expected recovery rate for that name, $\mathbb{E}(R_i)$, multiplied by some state dependent factor. For the model to make sense, we must have the restriction given on the final line of (A.4). The expected recovery rates, $\mathbb{E}(R_i)$, are input and these should be used during CDS bootstrapping. Further one must keep in mind that recovery rates fall in the $[0, 1]$ interval, thus one can not choose the multiplication factors too large. Other specification of the recovery rate are possible, but this is particularly straightforward to use.

A four state model is considered where the parameters are chosen as follows:

Parameter	State 1	State 2	State 3	State 4
p_j	30%	30%	30%	10%
α_j	0%	5%	10%	60%
β_j	1.3	1.2	0.7	0.4

The choice of the parameters result in a four state model, where low correlation goes accompanied by high recovery rates.

A.4 Random Factor Loadings

In Andersen and Sidenius (2004) a random factor loadings model is discussed. In their paper it is assumed that the correlation is not only a random number, but in addition it depends on the realisation of the common factor. This setup can capture scenarios of a bad state of the economy accompanied by high correlations, a phenomenon regularly observed in practice. Their research shows that a simple relationship between the correlation parameter and the common factor can create plausible Base Correlation skews. In it simplest form a flat correlation is used which can take on two values. One large value, leading to a large amount of correlation and one small value. This model specifies the copula as follows:

$$\begin{aligned} X_i &\equiv \rho \cdot Y + v_i \cdot \xi_i + m_i \\ \rho &= \begin{cases} \alpha, & Y \leq \theta \\ \beta, & Y > \theta \end{cases} \end{aligned} \tag{A.5}$$

Here, the common factor is denoted by Y, the latent variables by X_i. The term a is the factor loading and determines the correlation. As can be seen the correlation parameter is either α, or β, depending on the value of the common factor. The terms v_i and m_i in the first line are chosen such that the latent variables X_i have expectation equal to zero and variance equal to 1. In case one chooses $\alpha > \beta$ correlation will be larger in case the economy is in a bad state. More information can be found in Andersen and Sidenius (2004).

Here we only consider the simplest for of the model, given by formula (A.5). The following parameter choice is made:

α	β	θ
$\sqrt{0.60}$	$\sqrt{0.05}$	-2.6

Thus in case the economy is in a bad state, i.e. when the realisation of the common factor is smaller than -2.6, the correlation between the latent variables of the names in the basket is 60%, otherwise it is only at 5%.

A.5 Double t-copula

In Hull and White (2004) a t-distribution is used to model both the distribution of the common factor as well as those of the idiosyncratic factors. For large degrees of freedom a student's t-distribution will closely resemble a standard normal distribution and thus the standard model, given in (A.1) can be seen as a special case of the Double t-Copula. It is assumed that the latent variables for all names are identically distributed according to the following model:

$$X_i \equiv \alpha\sqrt{\frac{\nu_Y - 2}{\nu_Y}}Y + \sqrt{1-\alpha^2}\sqrt{\frac{\nu_\varepsilon - 2}{\nu_\varepsilon}}\xi_i \tag{A.6}$$

Here the terms ν_Y and ν_ε denote the degrees of freedom of the common factor and the idiosyncratic term, respectively. The scaling is done to guarantee unit variance. One of the main drawbacks of this model is that its implementation is computationally intensive. Numerical routines are required for the cumulative distribution, out-integration of the common factor and determination of thresholds.

The following table lists the choices for the three parameters:

α	ν_Y	ν_ε
$\sqrt{0.05}$	3.0	2.1

Thus a correlation parameter of 5% is used in combination with a low number for the degrees of freedom, for both the common factor as well as the idiosyncratic terms.

A.6 External Defaults Model

A third alternative model that is considered is the external defaults model, given in Voort (2005). For a given positive correlation, a fast default event of a certain name will have a large influence on the default likelihood of the surviving names. However when a fast default event occurs, which was totally unexpected, it is likely that such a default event is not caused by economical reasons, but rather by external reasons, such as fraud. This idea is captured in the external defaults model:

$$\begin{aligned} X_i &\equiv \rho \cdot Y + \sqrt{1-\rho^2}\cdot \xi_i \\ \tau_i \le t &\Leftrightarrow \min(X_i, Z_i) \le \chi_i(t) \end{aligned} \tag{A.7}$$

The second line of this formula describes that the default event is either caused by economic circumstances, when $X_i < Z_i$, or has an external cause when $Z_i < X_i$. The default barrier, $\chi_i(t)$, should be chosen such that the marginal default distributions are matched, as is done under the standard model. The external factor Z_i can be given any distribution. In case one chooses a distribution with a fatter tail than that of the standard normal distribution of X_i, the model has an attractive feature. This would mean that a default event which occurs against all expectations, for example a default event of a high grade name within a single month, is likely to have an external cause. Using a simple normal distribution for Z_i will result in upward sloping Base Correlation skew, though be it somewhat concave.

The following table lists the choices for the three parameters:

ρ	μ	σ
$\sqrt{0.8}$	1.5	2.1

Thus, without considering external defaults, the correlation between the latent variables X_i is 80%. The inclusion of the external defaults will bring this number down.

B DJ CDX Base Correlation Skews

In the following figures the 5Y and 10Y Base Correlation skews implied by the models are presented. In addition the Base Correlation skew as implied by the market has been plotted.

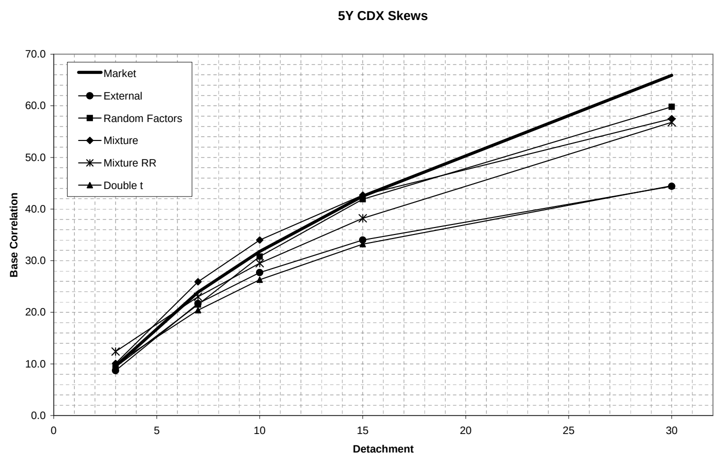

Figure 5: Market and model implied Base Correlation skews for DJ CDX 5Y. Different one factor copula models are considered. More details on the models and the parameter choices can be found in appendix A.

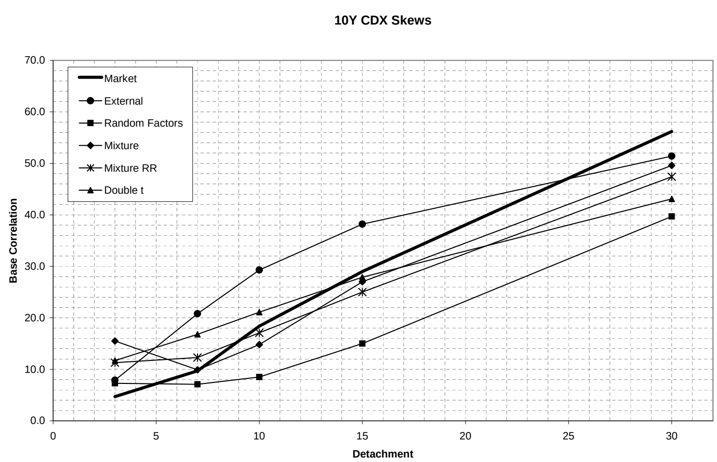

Figure 6: Market and model implied Base Correlation skews for DJ CDX 10Y. Different one factor copula models are considered. More details on the models and the parameter choices can be found in appendix A.